

# Uniqueness for an inverse problem for a semilinear time-fractional diffusion equation

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## Abstract

An inverse problem to determine a space-dependent factor in a semilinear time-fractional diffusion equation is considered. Additional data are given in the form of an integral with the Borel measure over the time. Uniqueness of the solution of the inverse problem is studied. The method uses a positivity principle of the corresponding differential equation that is also proved in the paper.

## Introduction

Anomalous diffusion processes in porous, fractal, biological etc. media are described by differential equations containing fractional derivatives. Depending on the nature of the process, the model may involve either fractional time or fractional space derivatives or both ones [6, 19, 31].

In many practical situations the properties of the medium or sources are unknown and they have to be reconstructed solving inverse problems [3, 11, 24, 32]. Then some additional information for the solution of the differential equation is needed to recover the unknowns. If space-dependent quantities are to be determined, such additional conditions may involve instant (e.g. final) measurements or integrated measurements over time.

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In case space and time variables are separable, the solution of the time-fractional diffusion equation can be expressed by a formula that is deduced by means of the Fourier expansion with respect to the space variables and integration using Mittag-Leffler functions with respect to the time variable. This formula enables to prove uniqueness of reconstruction of space-dependent factors of source terms from final overdetermination [4, 13, 27]. However, this method fails even in the linear case when the variables are not separable.

In the present paper we prove the uniqueness for an inverse problem to determine a space-dependent factor in the time-fractional diffusion equation in a more general case when the equation may be semilinear. Additional condition is given in a form of an integral with Borel measure over the time that includes as a particular case the final overdetermination. Such an inverse problem has possible applications in modelling of fractional reaction-diffusion processes [5, 12, 23], more precisely in reconstruction of certain parameters of inhomogeneous media.

Our results are global in time, but contain certain cone-type restrictions that may depend on a time interval. We will adjust a method that was applied to inverse problems for usual parabolic equations [8, 9] and generalized to an integrodifferential case [10] and parabolic semilinear case [1]. The method is based on positivity principles of solutions of differential equations that follow from extremum principles. Extremum principles for time-fractional diffusion equations are proved in the linear case in [2, 16, 22] and in a nonlinear divergence-type case in [29]. But these results are not directly applicable in our case, because of the lack of the semilinear term. Therefore, we prove an independent positivity principle in our paper.

To the authors' opinion, such a positivity principle has a scientific value independently of the inverse problem, too. States of several reaction-diffusion models are positive functions, e.g. probability densities [12].

The stability of the solution of the inverse problem will not be studied in this paper. In the linear case the stability follows from the uniqueness by means of the Fredholm alternative [24]. The solution continuously depends on certain derivatives of the data, hence the problem is moderately ill-posed.

The plan for the paper is as follows. In the first section we formulate the direct and inverse problems. Second section contains auxiliary results about a linear direct problem that are necessary for the analysis of the inverse problem. Third section is devoted to the positivity principle. In the fourth section we prove the main uniqueness theorem in case of the general additional condition involving the Borel measure. The next section contains a particular uniqueness result in a case of the Lebesgue measure with a weight. In the last section we discuss some crucial assumptions of the uniqueness theorems.

# 1 Problem formulation

Let  $\Omega \subset \mathbb{R}^n$  be a bounded open domain with a sufficiently smooth boundary  $\partial\Omega$ . We consider the semilinear fractional diffusion equation

$$D_t^\beta[u(t, x) - u_0(x)] = A(x)u(t, x) + f(u(t, x), t, x), \quad t \in (0, T), \quad x \in \Omega, \quad (1.1)$$

with the initial and boundary conditions

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (1.2)$$

$$\mathcal{B}u(t, x) = g(t, x), \quad (t, x) \in (0, T) \times \partial\Omega \quad (1.3)$$

where

$$\text{either (I) } \mathcal{B}u = u \quad \text{or} \quad \text{(II) } \mathcal{B}u = \omega \cdot \nabla u \quad (1.4)$$

with some  $x$ -dependent vector function  $\omega(x) = (\omega_1(x), \dots, \omega_n(x))$  such that  $\omega(x) \cdot \nu(x) > 0$  where  $\nu$  is the outer normal to  $\partial\Omega$ . Here and in the sequel  $\nabla$  denotes the gradient operator with respect to the space variables. Moreover,

$$D_t^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} v(\tau) d\tau$$

is the Riemann-Liouville fractional derivative of order  $\beta \in (0, 1)$  and the operator  $A$  has the form

$$A(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j},$$

where the principal part is uniformly elliptic, i.e.

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq c |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad x \in \Omega \quad \text{for some } c > 0.$$

Note that in case of sufficiently smooth  $u$  the term in the left-hand side of (1.1) is actually the Caputo derivative of  $u$ , i.e.  $D_t^\beta[u(t, x) - u_0(x)] = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} u_\tau(\tau, x) d\tau$ . The conditions (1.1), (1.2) and (1.3) form a direct problem for the function  $u$ .

The existence and uniqueness of the solution of (1.1) - (1.3) in case  $f(w, t, x)$  depends only on  $w$  are proved in [14, 17]. More precisely, [17] considers the problem with Dirichlet boundary conditions and [14] the one-dimensional problem with Neumann boundary conditions. In the appendix of the paper we prove independent existence and uniqueness theorems for (1.1) - (1.3) in the general form.

We will consider the case when the nonlinearity function  $f$  has the following form:

$$f(w, t, x) = a(w, t, x)z(x) + b(w, t, x), \quad (1.5)$$

where  $a$  and  $b$  are given but the factor  $z$  is unknown.

Let us formulate an inverse problem.

IP. Let  $\mu$  be a positive Borel measure such that  $\text{supp}(\mu) \cap (0, T] \neq \emptyset$ . Determine a pair of functions  $(z, u)$  such that the conditions (1.1), (1.2), (1.3), (1.5) and the additional condition

$$\int_0^T u(t, x) d\mu = d(x), \quad x \in \Omega \quad (1.6)$$

with some given function  $d$  is satisfied.

We note that a special case of  $\mu$  is the Dirac measure concentrated at the final moment  $t = T$ . Then the condition (1.6) reads  $u(T, x) = d(x)$ ,  $x \in \Omega$ .

## 2 Preliminaries

In this section we formulate and prove some auxiliary results. In addition to  $D_t^\beta$ , we introduce the operator of fractional integration of order  $\gamma > 0$ :

$$J_t^\gamma w(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma-1} w(\tau) d\tau.$$

In the sequel we consider the fractional differentiation and integration in Bessel potential and Hölder spaces. We consider the abstract Bessel potential spaces  $H_p^\beta([0, T]; X)$  for Banach spaces  $X$  of the class  $\mathcal{HT}$  that is defined in the following manner ([28], p. 18, [21], p. 216):<sup>1</sup>

$$\mathcal{HT} = \{X : X - \text{Banach space, the Hilbert transform is bounded in } L^q(\mathbb{R}; X) \text{ for some } q \in (1, \infty)\}$$

Due to an embedding theorem, it holds  $H_p^\beta([0, T]; X) \subset C^\alpha([0, T]; X)$  provided  $\frac{1}{\beta} < p < \infty$ ,  $\alpha \in (0, \beta - \frac{1}{p})$  and  $X$  is of the class  $\mathcal{HT}$ .

Let us formulate four lemmas that directly follow from known results in the literature.

**Lemma 1.** *Let  $X$  be a Banach space of the class  $\mathcal{HT}$  and  $\frac{1}{\beta} < p < \infty$ . If  $w \in$*

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<sup>1</sup>For example,  $L_s(\Omega)$ ,  $1 < s < \infty$ , are of the class  $\mathcal{HT}$ .

$H_p^\beta([0, T]; X)$  and  $w(0) = 0$  then  $D_t^\beta w \in L_p((0, T); X)$ . If  $w \in L_p((0, T); X)$  then  $J_t^\beta w \in H_p^\beta([0, T]; X)$  and  $J_t^\beta w(0) = 0$ . Moreover, the relations

$$J_t^\beta D_t^\beta (w - w(0)) = w - w(0) \quad \forall w \in H_p^\beta([0, T]; X), \quad (2.1)$$

$$D_t^\beta J_t^\beta w = w \quad \forall w \in L_p((0, T); X) \quad (2.2)$$

are valid.

*Proof.* These assertions follow from arguments presented in [28], p. 28, 29.

**Lemma 2.** *Let  $X$  be a Banach space. If  $w \in C^\alpha([0, T]; X)$ ,  $\alpha \in (0, 1)$ ,  $w(0) = 0$  and  $\beta \in (0, 1 - \alpha)$  then  $J_t^\beta w \in C^{\alpha+\beta}([0, T]; X)$  and  $J_t^\beta w(0) = 0$ . If  $w \in C^\alpha([0, T]; X)$ ,  $\alpha \in (0, 1)$ ,  $w(0) = 0$  and  $\beta \in (0, \alpha)$  then  $D_t^\beta w \in C^{\alpha-\beta}([0, T]; X)$  and  $D_t^\beta w(0) = 0$ .*

*Proof.* Assertions of the lemma follow from Thms 14, 19 and the first part of Thm. 20 of [7] if we continue  $w(t)$  by zero for  $t < 0$ . Although [7] considers the case  $X = \mathbb{R}$ , all arguments included in proofs of the mentioned theorems automatically hold in case of arbitrary Banach space  $X$ , too.  $\square$

**Lemma 3** *Let  $\frac{1}{\beta} < p < \infty$ ,  $p \notin \{\frac{1}{\beta} + \frac{1}{2}; \frac{2}{\beta} + 1\}$ . Assume  $a_{ij} \in C(\overline{\Omega})$ ,  $a_j \in L_\infty(\Omega)$ ,  $q \in L_\infty((0, T) \times \Omega)$ ,  $\omega \in (C^r(\partial\Omega))^n$ ,  $r > 1 - \frac{1}{p}$ . Then the problem*

$$w(t, x) = J_t^\beta (A(x) + q(t, x))w(t, x) + \varphi(t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (2.3)$$

$$\mathcal{B}w(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega \quad (2.4)$$

has unique solution

$$w \in U_p := H_p^\beta([0, T]; L_p(\Omega)) \cap L_p((0, T); H_p^2(\Omega)) \quad (2.5)$$

such that  $w(0, \cdot) = 0$  if and only if  $\varphi \in \Phi_p = \{\varphi \in H_p^\beta([0, T]; L_p(\Omega)) : \varphi(0, \cdot) = 0\}$ . The operator  $\mathcal{S}$  that maps  $\varphi$  to  $w$  is continuous from the space  $\Phi_p$  to the space  $U_p$ .

*Proof.* The existence and uniqueness assertions of Lemma 3 are a particular case of a more general maximal regularity result proved in Thm 4.3.1 of [28]. The proof of the cited theorem is based on approximation of the problem by a sequence of localized problems with constant coefficients. For the localized problems, existence and uniqueness theorem for an abstract parabolic evolutionary integral equation containing unbounded operator is applied. The study of the such an abstract equation is based on the construction of a solution in the form of a variation of parameters formula that contains a convolution of a resolvent (operator) of the equation with the given right-hand side. The desired assertions follow from the properties of the resolvent. A formula of the resolvent is constructed and its properties established by means of the Laplace transform. The analysis of the mentioned abstract parabolic

equation is contained also in [30].  $\square$

**Lemma 4** *Let  $X$  be a Banach space and  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  a closed densely defined unbounded operator satisfying the following property:*

$$\rho(\mathcal{A}) \supset \Sigma(\beta\pi/2), \quad \exists M > 0 : \|(\lambda - \mathcal{A})^{-1}\| \leq \frac{M}{|\lambda|} \quad \forall \lambda \in \Sigma(\beta\pi/2) \quad (2.6)$$

where  $\rho(\mathcal{A})$  is the resolvent set of  $\mathcal{A}$  and  $\Sigma(\theta) = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$ . Let  $X_{\mathcal{A}}$  be the domain of  $\mathcal{A}$  endowed with the graph norm  $\|z\|_{X_{\mathcal{A}}} = \|z\| + \|\mathcal{A}z\|$ . If  $\varphi \in C^\alpha([0, T]; X)$  for some  $\alpha \in (0, 1)$  and  $\varphi(0) = 0$  then the equation

$$w(t) = \mathcal{A}J_t^\beta w(t) + \varphi(t), \quad t \in [0, T]$$

has a solution  $w \in C^\alpha([0, T]; X)$  satisfying the properties  $w(0) = 0$ ,  $J_t^\beta w \in C^\alpha([0, T]; X_{\mathcal{A}})$ . The operator  $\mathcal{Q}_\alpha$  that maps  $\varphi$  to  $w$  is continuous from the space  $\{\varphi : \varphi \in C^\alpha([0, T]; X), \varphi(0) = 0\}$  to the space  $C^\alpha([0, T]; X)$ . If, additionally,  $\varphi \in C^\alpha([0, T]; X_{\mathcal{A}})$  then  $w \in C^\alpha([0, T]; X_{\mathcal{A}})$ .

*Proof.* Lemma 4 follows from Thm. 2.4, the estimate (2.31) in the proof of Thm. 2.4 and Example 2.1 of [21]. The idea of the proof is analogous to the proof of previous lemma. The solution is constructed in a form of a convolution that contains a resolvent operator and properties of the resolvent are established by means of the Laplace transform.  $\square$

Now we prove an existence theorem in Hölder spaces for the direct problem (1.1), (1.2), (1.3) in the linear case. This result will be used in the analysis of IP.

**Theorem 1.** *Let  $f(w, t, x) = q(t, x)w + \psi(t, x)$ ,  $u_0 = 0$ ,  $g = 0$ . Assume that  $a_{ij}, a_j \in C^\alpha(\overline{\Omega})$ ,  $\omega \in (C^{1+\alpha}(\partial\Omega))^n$  and*

$$\begin{aligned} \psi &\in C^{\alpha+\beta}([0, T]; L_p(\Omega)) \cap C^\alpha([0, T]; C^\alpha(\overline{\Omega})), \quad \psi(0, \cdot) = 0, \\ q &\in C^{\alpha+\beta}([0, T]; L_\infty(\Omega)) \cap C^\alpha([0, T]; C^\alpha(\overline{\Omega})) \end{aligned}$$

with some  $\alpha \in (0, 1)$  and  $p \in (\max\{\frac{n}{2}, \frac{1}{\beta}\}, \infty)$ ,  $p \notin \{\frac{1}{\beta} + \frac{1}{2}, \frac{2}{\beta} + 1\}$ . Then there exists  $\alpha_1 \in (0, 1 - \beta)$  such that the problem (1.1), (1.2), (1.3) has a solution

$$u \in C^{\alpha_1+\beta}([0, T]; C^{\alpha_1}(\overline{\Omega})) \cap C^{\alpha_1}([0, T]; C^{2+\alpha_1}(\overline{\Omega})) \quad (2.7)$$

with  $D_t^\beta u \in C^{\alpha_1}([0, T]; C^{\alpha_1}(\overline{\Omega}))$ . The solution is unique in the wider space  $U_p$ . If in addition,

$$\psi_t \in L_p((0, T); L_p(\Omega)), \quad q_t \in L_p((0, T); L_p(\Omega)) \cap L_1((0, T); L_\infty(\Omega))$$

then  $u_t \in U_p$  and  $u_t(0, \cdot) = 0$ .

*Proof.* Note that the problem (1.1), (1.2), (1.3) is in the space  $U_p$  equivalent to the following problem:

$$u(t, x) = J_t^\beta (A(x) + q(t, x))u(t, x) + J_t^\beta \psi(t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (2.8)$$

$$\mathcal{B}u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega. \quad (2.9)$$

By Lemma 3, the latter problem has a unique solution  $u \in U_p$ . This proves the existence and uniqueness assertions in  $U_p$ .

Next we are going to prove the inclusion (2.7). According to arguments of Section 3.1.1 and Thm 3.1.3 of [18], there exists  $\xi \in \mathbb{R}$  such that the operator  $\mathcal{A} = A + \xi$  with the domain  $D(\mathcal{A}) = \{z \in W_p^2(\Omega) : \mathcal{B}z|_{\partial\Omega} = 0\}$  in  $X = L_p(\Omega)$  satisfies the following conditions:

$$\rho(\mathcal{A}) \supset \Sigma(\pi/2), \quad \exists M > 0 : \|(\lambda - \mathcal{A})^{-1}\| \leq \frac{M}{|\lambda|} \quad \forall \lambda \in \Sigma(\pi/2).$$

Since  $0 < \beta < 1$ , this relation implies (2.6). Moreover,  $\mathcal{A}$  is closed and densely defined in  $X$ . Thus,  $\mathcal{A}$  satisfies the assumptions of Lemma 4. Let us consider the problem

$$v(t, x) = (A(x) + \xi)J_t^\beta v(t, x) + \varphi(t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (2.10)$$

$$\mathcal{B}v(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega \quad (2.11)$$

with  $\varphi(t, x) = (q(t, x) - \xi)u(t, x) + \psi(t, x)$ . Observing that  $u \in U_p$  and  $U_p \subset C^{\alpha_2}([0, T]; L_p(\Omega))$  with  $\alpha_2 \in (0, \beta - \frac{1}{p})$ , by the embedding theorem, as well as the assumptions of the theorem we have  $\varphi \in C^{\alpha_2}([0, T]; L_p(\Omega))$  and  $\varphi(0, x) = 0$ . Further, choosing some number  $\alpha_3 \in (0, \alpha_2] \cap (0, 1 - \beta)$  we also have  $\varphi \in C^{\alpha_3}([0, T]; L_p(\Omega))$ . Applying Lemma 4 to the problem (2.10), (2.11), we conclude that it has a solution  $v \in C^{\alpha_3}([0, T]; L_p(\Omega))$ . Taking the operator  $J_t^\beta$  from the relations (2.10), (2.11) and subtracting from (2.8), (2.9) we see that the function  $w = u - J_t^\beta v$  solves the problem (2.3), (2.4) with  $\varphi = 0$ . Due to the uniqueness it holds  $w = 0$ , hence  $u = J_t^\beta v$ . By the proved relation  $v \in C^{\alpha_3}([0, T]; L_p(\Omega))$  and Lemma 2 we get  $u \in C^{\alpha_3+\beta}([0, T]; L_p(\Omega))$ .

According to the latter relation and assumptions of the theorem we can improve properties of  $\varphi$ . Namely, it holds  $\varphi \in C^{\alpha_4+\beta}([0, T]; L_p(\Omega))$  with  $\alpha_4 = \min\{\alpha_3; \alpha\} \in (0, 1 - \beta)$ . Applying Lemma 4 again to (2.10), (2.11), we obtain  $v \in C^{\alpha_4+\beta}([0, T]; L_p(\Omega))$  with  $J_t^\beta v \in C^{\alpha_4+\beta}([0, T]; W_p^2(\Omega))$ . From the latter inclusion and the embedding theorem we deduce

$$u = J_t^\beta v \in C^{\alpha_4+\beta}([0, T]; C^\gamma(\overline{\Omega})) \quad (2.12)$$

with some  $\gamma > 0$ . Lemma 2 implies  $D_t^\beta u \in C^{\alpha_4}([0, T]; C^\gamma(\overline{\Omega}))$ .

Further, let us rearrange the terms in (1.1), (1.2), (1.3) to get the following family of elliptic problems:

$$A(x)u(t, x) = \phi(t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (2.13)$$

$$\mathcal{B}u(t, x) = 0, \quad x \in \partial\Omega, \quad t \in (0, T) \quad (2.14)$$

with  $\phi(t, x) = D_t^\beta u(t, x) - q(t, x)u(t, x) - \psi(t, x)$ . According to the proved properties of  $u$  and the assumptions of theorem we have  $\phi \in C^{\alpha_4}([0, T]; C^{\alpha_5}(\overline{\Omega}))$  with  $\alpha_5 = \min\{\gamma; \alpha\}$ . Using the well-known Schauder-estimates for the elliptic problems (e.g. [15], Ch. III) and taking the Hölder-estimates with respect to  $t$  we obtain  $u \in C^{\alpha_4}([0, T]; C^{2+\alpha_5}(\overline{\Omega}))$ . Combining this relation with (2.12) we prove the assertion (2.7) with  $\alpha_1 = \min\{\alpha_4; \alpha_5\}$ .

To prove the additional assertion, let us consider the problem

$$u_1(t, x) = J_t^\beta(A(x) + q(t, x))u_1(t, x) + \phi_1(t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (2.15)$$

$$\mathcal{B}u_1(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega \quad (2.16)$$

with  $\phi_1 = J_t^\beta(q_t u + \psi_t)$ . Due to the proven properties of  $u$  and the additional assumptions of  $\psi, q$  we have  $q_t u + \psi_t \in L_p((0, T); L_p(\Omega))$ , hence  $\phi_1 \in H_p^\beta([0, T]; L_p(\Omega))$ ,  $\phi_1(0, x) = 0$ . Lemma 3 implies that the problem (2.15), (2.16) has a unique solution  $u_1 \in U_p$  and  $u_1(0, \cdot) = 0$ . The next aim is to show that  $u_t = u_1$ . This will complete the proof.

Taking the operator  $J_t^1$  from the relations (2.15), (2.16) and subtracing from (2.8), (2.9) we see that  $w = u - J_t^1 u_1$  satisfies the following problem:

$$w(t, x) = J_t^\beta(A(x) + q(t, x))w(t, x) - J_t^{1+\beta}(q_t w), \quad x \in \Omega, \quad t \in (0, T), \quad (2.17)$$

$$\mathcal{B}w(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega. \quad (2.18)$$

Define  $(\mathcal{P}_t z)(\tau, \cdot) = \begin{cases} z(\tau, \cdot) & \text{if } \tau \leq t \\ 0 & \text{if } \tau > t. \end{cases}$  Evidently, the relation  $(\mathcal{S}J_t^\beta z)(\tau, \cdot) =$

$(\mathcal{S}J_t^\beta \mathcal{P}_t z)(\tau, \cdot)$  is valid for  $\tau \leq t$  and any  $z \in L_p((0, T); L_p(\Omega))$ , where  $\mathcal{S}$  is the operator defined in Lemma 3. Let us fix  $t \in (0, T)$ . Applying Lemma 3 to the problem (2.17), (2.18) and using the embedding theorem we deduce the estimate

$$\begin{aligned} \max_{0 \leq \tau \leq t} \|w(\tau, \cdot)\|_{L_p(\Omega)} &= \max_{0 \leq \tau \leq t} \|\mathcal{S}J_t^\beta \mathcal{P}_t J_t^1(q_t w)(\tau, \cdot)\|_{L_p(\Omega)} \\ &\leq \tilde{C}_1 \|\mathcal{S}J_t^\beta \mathcal{P}_t J_t^1(q_t w)\|_{U_p} \leq \tilde{C}_1 \|\mathcal{S}\| \|J_t^\beta \mathcal{P}_t J_t^1(q_t w)\|_{H_p^\beta([0, T]; L_p(\Omega))} \\ &\leq \tilde{C}_2 \|\mathcal{P}_t J_t^1(q_t w)\|_{L_p((0, T); L_p(\Omega))} = \tilde{C}_2 \|J_t^1(q_t w)\|_{L_p((0, t); L_p(\Omega))} \\ &\leq \tilde{C}_2 \left[ \int_0^t \left[ \int_0^\tau \|(q_t w)(s, \cdot)\|_{L_p(\Omega)} ds \right]^p d\tau \right]^{\frac{1}{p}} \\ &\leq \tilde{C}_2 \|q_t\|_{L_1((0, T); L_\infty(\Omega))} \left[ \int_0^t \left( \max_{0 \leq s \leq \tau} \|w(s, \cdot)\|_{L_p(\Omega)} \right)^p d\tau \right]^{\frac{1}{p}}, \end{aligned} \quad (2.19)$$



where  $\tilde{C}_1$  and  $\tilde{C}_2$  are some constants. Thus, we have proved the inequality  $\left(\max_{0 \leq \tau \leq t} \|w(\tau, \cdot)\|_{L_p(\Omega)}\right)^p - \tilde{C}_3 \int_0^t \left(\max_{0 \leq s \leq \tau} \|w(s, \cdot)\|_{L_p(\Omega)}\right)^p d\tau \leq 0$  for any  $t \in (0, T)$  with some constant  $\tilde{C}_3$ . Making use of the Gronwall's theorem we obtain  $\max_{0 \leq \tau \leq t} \|w(\tau, \cdot)\|_{L_p(\Omega)} = 0$  for any  $t \in (0, T)$ . This yields  $w = u - J_t^1 u_1 = 0$  and in turn the desired equality  $u_t = u_1$ . Theorem 1 is proved.  $\square$

### 3 Positivity principle

In this section we prove a positivity principle for the solution of the equation (1.1) in a bit more general form. Namely, we consider the equation

$$D_t^{\{k\}}[u(t, x) - u_0(x)] = A(x)u(t, x) + f(u(t, x), t, x), \quad t \in (0, T), \quad x \in \Omega, \quad (3.1)$$

where

$$D_t^{\{k\}}v = \frac{d}{dt}k * v, \quad (k * v)(t) = \int_0^t k(t - \tau)v(\tau)d\tau$$

and the kernel  $k$  has the following properties:

$$k \in L_1(0, T) \cap C(0, T], \quad k > 0, \quad k - \text{decreasing}, \quad (3.2)$$

$$k(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow 0^+. \quad (3.3)$$

This generalization doesn't make the proofs more complicated. Moreover, some parts of proofs even require such a more general treatment.

**Theorem 2.** Assume (3.2), (3.3),  $a_{ij}, a_j \in C(\overline{\Omega})$ ,  $\omega \in (C(\partial\Omega))^n$ ,

$$f \in C(\mathbb{R} \times [0, T] \times \overline{\Omega}), \quad (3.4)$$

$$\exists M \geq 0, \eta > 0 : f(w, t, x) \geq -M|w| \quad \text{in} \quad (-\eta, 0) \times [0, T] \times \overline{\Omega}. \quad (3.5)$$

Let  $u \in C([0, T] \times \overline{\Omega})$  with  $u_{x_j}, u_{x_i, x_j} \in C((0, T] \times \overline{\Omega})$ ,

$$D_t^{\{k\}}(u - u_0) \in C((0, T] \times \overline{\Omega}) \quad (3.6)$$

solves the problem (3.1), (1.2), (1.3). Moreover, let

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^\epsilon k(\tau)d\tau \sup_{0 \leq s \leq \epsilon} |u(t - s, x) - u(t, x)| = 0 \quad \forall t \in (0, T], \quad x \in \overline{\Omega}. \quad (3.7)$$

Finally, we assume  $u_0 \geq 0$  and  $g \geq 0$ . Then the following assertions are valid:

- (i)  $u \geq 0$ ;

(ii) if  $u(t_0, x_0) = 0$  at some point  $(t_0, x_0) \in (0, T] \times \Omega_N$ , where

$$\Omega_N = \begin{cases} \Omega & \text{in case I} \\ \overline{\Omega} & \text{in case II,} \end{cases}$$

then  $u(t, x_0) = 0$  for any  $t \in [0, t_0]$ .

Recall that the cases I and II were defined in (1.4).

**Remark 1.** In case  $k$  is the kernel of the fractional derivative  $D_t^\beta$ , i.e.  $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$ ,  $0 < \beta < 1$ , the assumptions (3.6), (3.7) are satisfied provided  $u \in C^{\beta'}([0, T]; C(\overline{\Omega}))$  with some  $\beta' > \beta$ .

Before proving Theorem 2, we state and prove a lemma.

**Lemma 5** (a minimum principle). *Let the assumptions of Theorem 2 be satisfied, except for the conditions (3.3) and  $u_0, g \geq 0$ . Moreover, let instead of (3.5) the following stronger condition*

$$f(w, t, x) \geq 0 \quad \text{in } (-\infty, 0) \times [0, T] \times \overline{\Omega} \quad (3.8)$$

*be satisfied. Let  $(t_1, x_1)$  be the minimum point of a solution of (3.1), (1.2) over  $[0, T] \times \overline{\Omega}$ . If this is a stationary minimum with respect to  $x$ , i.e.  $\nabla u(t_1, x_1) = 0$ , then  $u(t_1, x_1) \geq \min_{x \in \overline{\Omega}} \{0; u_0(x)\}$ .*

*Proof.* Suppose that the assertion of the lemma doesn't hold. Then

$$(t_1, x_1) \in (0, T] \times \overline{\Omega} \quad \text{and} \quad u(t_1, x_1) < 0, \quad u(t_1, x_1) < \min_{x \in \overline{\Omega}} u_0(x).$$

Since  $x = x_1$  is the stationary minimum point of  $u(t_1, x)$  over  $\overline{\Omega}$  and the principal part of  $A$  is elliptic, it holds  $\sum_{i,j=1}^n a_{ij}(x_1) u_{x_i x_j}(t_1, x_1) \geq 0$  (see [20]). Thus,

$$A(x_1)u(t_1, x_1) \geq 0. \quad (3.9)$$

Further, in view of (3.8) and  $u(t_1, x_1) < 0$  we get

$$f(u(t_1, x_1), t_1, x_1) \geq 0. \quad (3.10)$$

Next let us study the term  $D_t^{\{k\}}[u(t, x) - u_0(x)]$  at  $t = t_1$  and  $x = x_1$ . We have

$$\begin{aligned} D_t^{\{k\}}[u(t, x_1) - u_0(x_1)] \Big|_{t=t_1} &= \frac{\partial}{\partial t} k * [u(t, x_1) - u_0(x_1)] \Big|_{t=t_1} \\ &= \lim_{\epsilon \rightarrow 0+} \frac{1}{\epsilon} \left\{ \int_0^{t_1} k(t_1 - \tau) [u(\tau, x_1) - u_0(x_1)] d\tau \right. \\ &\quad \left. - \int_0^{t_1 - \epsilon} k(t_1 - \epsilon - \tau) [u(\tau, x_1) - u_0(x_1)] d\tau \right\}. \end{aligned} \quad (3.11)$$

Observing that  $u(\tau, x_1) \geq u(t_1, x_1)$  holds for  $0 \leq \tau \leq t_1$  and taking the relation (3.2) into account we estimate the term between the brackets  $\{\}$  in (3.11):

$$\begin{aligned}
& \int_0^{t_1} k(t_1 - \tau)[u(\tau, x_1) - u_0(x_1)]d\tau \\
& - \int_0^{t_1 - \epsilon} k(t_1 - \epsilon - \tau)[u(\tau, x_1) - u_0(x_1)]d\tau \\
& = \int_0^{t_1 - \epsilon} [k(t_1 - \tau) - k(t_1 - \epsilon - \tau)][u(\tau, x_1) - u_0(x_1)]d\tau \\
& + \int_{t_1 - \epsilon}^{t_1} k(t_1 - \tau)[u(\tau, x_1) - u_0(x_1)]d\tau \\
& \leq \int_0^{t_1 - \epsilon} [k(t_1 - \tau) - k(t_1 - \epsilon - \tau)]d\tau [u(t_1, x_1) - u_0(x_1)] \\
& + \int_{t_1 - \epsilon}^{t_1} k(t_1 - \tau)[u(\tau, x_1) - u_0(x_1)]d\tau \\
& = \int_{t_1 - \epsilon}^{t_1} k(\tau)d\tau [u(t_1, x_1) - u_0(x_1)] + \int_0^\epsilon k(\tau)[u(t_1 - \tau, x_1) - u(t_1, x_1)]d\tau.
\end{aligned}$$

Thus, from (3.11) due to (3.7), (3.2) and the inequalities  $t_1 > 0$ ,  $u(t_1, x_1) < u_0(x_1)$  we obtain

$$\begin{aligned}
D_t^{\{k\}}[u(t, x_1) - u_0(x_1)] \Big|_{t=t_1} & \leq \lim_{\epsilon \rightarrow 0+} \frac{1}{\epsilon} \left[ \int_{t_1 - \epsilon}^{t_1} k(\tau)d\tau [u(t_1, x_1) - u_0(x_1)] \right. \\
& \left. + \int_0^\epsilon k(\tau)[u(t_1 - \tau, x_1) - u(t_1, x_1)]d\tau \right] \\
& = k(t_1) [u(t_1, x_1) - u_0(x_1)] < 0.
\end{aligned} \tag{3.12}$$

The inequalities (3.12), (3.9) and (3.10) show that the left-hand side of the equation (3.1) is negative but the right-hand side is nonnegative at  $t = t_1$ ,  $x = x_1$ . This is a contradiction. Therefore, the assertion of the lemma is valid.  $\square$

*Proof of Theorem 2.* Firstly, let us prove (i) in case (3.5) is replaced by the stronger condition (3.8). Let again  $(t_1, x_1)$  be the minimum point of  $u$  over  $[0, T] \times \overline{\Omega}$ . Suppose that (i) doesn't hold. Then  $u(t_1, x_1)$  is negative. In case I the point  $(t_1, x_1)$  is contained in the subset  $(0, T] \times \Omega$ . This implies that  $\nabla u(t_1, x_1) = 0$  and in view of Lemma 5 we reach the contradiction. In case II the minimum point  $(t_1, x_1)$  is contained in  $(0, T] \times \overline{\Omega}$ . If  $x_1 \in \Omega$ , we reach the contradiction in a similar manner.

Thus, it remains to consider the case II with  $x_1 \in \partial\Omega$ . In the minimum point at the boundary the inequality  $\omega(x_1) \cdot \nabla u(t_1, x_1) \leq 0$  holds. This together with the assumption  $\omega \cdot \nabla u|_{x \in \partial\Omega} = g \geq 0$  implies  $\omega(x_1) \cdot \nabla u(t_1, x_1) = 0$ . Moreover,

$\tau \cdot \nabla u(t_1, x_1) = 0$ , where  $\tau$  is any tangential direction on  $\partial\Omega$  at  $x_1$ , because  $x = x_1$  is the minimum point of the  $x$ -dependent function  $u(t_1, x)$  over the set  $\partial\Omega$ . Putting these relations together we see that  $\nabla u(t_1, x_1) = 0$ . Again, we reach the contradiction with Lemma 5. The assertion (i) is proved in case (3.8).

Secondly, we prove (i) in the general case (3.5). Note that due to the continuity of  $f$ , the relation (3.5) can immediately be extended to  $f(w, t, x) \geq -\left(M + \frac{D_q}{\eta}\right)|w|$  in  $(-q, 0) \times [0, T] \times \overline{\Omega}$  where  $D_q = \max_{\substack{-q \leq w \leq 0 \\ (t, x) \in [0, T] \times \overline{\Omega}}} |f(w, t, x)|$  and  $q > 0$  is an arbitrary number. Let us define the new nonlinearity function:

$$\widehat{f}(w, t, x) = \begin{cases} f(w, t, x) & \text{in case } w \geq -Q \\ f(-Q, t, x) & \text{in case } w < -Q, \end{cases} \quad \text{where } Q = \max_{\substack{t \in [0, T] \\ x \in \overline{\Omega}}} |u(t, x)|.$$

Then

$$\widehat{f}(w, t, x) \geq -\widehat{M}|w| \quad \text{in } (-\infty, 0) \times [0, T] \times \overline{\Omega}, \quad (3.13)$$

where  $\widehat{M} = M + \frac{D_q}{\eta}$ . According to the definition of  $\widehat{f}$ , we can rewrite the equation for  $u$  (3.1) in the following form:

$$D_t^{\{k\}}[u(t, x) - u_0(x)] = A(x)u(t, x) + \widehat{f}(u(t, x), t, x), \quad t \in (0, T), \quad x \in \Omega. \quad (3.14)$$

Define  $\tilde{u}(t, x) = e^{-\sigma t}u(t, x)$ , where  $\sigma > 0$  is a number that we will specify later. Then  $\tilde{u}$  solves the following equation:

$$D_t^{\{\tilde{k}\}}[\tilde{u}(t, x) - u_0(x)] = A(x)\tilde{u}(t, x) + \tilde{f}(\tilde{u}(t, x), t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (3.15)$$

and satisfies the initial and boundary conditions

$$\tilde{u}(0, x) = u_0(x), \quad x \in \Omega, \quad \mathcal{B}\tilde{u}(t, x) = \tilde{g}(t, x), \quad (t, x) \in (0, T) \times \partial\Omega, \quad (3.16)$$

where

$$\begin{aligned} \tilde{k}(t) &= e^{-\sigma t}k(t) + \sigma \int_T^t e^{-\sigma s}k(s)ds, \quad \tilde{g}(t) = e^{-\sigma t}g(t), \\ \tilde{f}(w, t, x) &= e^{-\sigma t}\widehat{f}(e^{\sigma t}w, t, x) - \sigma w \int_0^T e^{-\sigma s}k(s)ds + \sigma u_0(x) \int_t^T e^{-\sigma s}k(s)ds. \end{aligned}$$

We are going to show that the assumptions of the theorem are satisfied for the problem (3.15), (3.16). The smoothness conditions  $\tilde{k} \in L_1(0, T) \cap C(0, T]$ ,  $\tilde{f} \in C(\mathbb{R} \times [0, T] \times \overline{\Omega})$ ,  $\tilde{u} \in C([0, T] \times \overline{\Omega})$ ,  $\tilde{u}_{x_j}, \tilde{u}_{x_i, x_j} \in C((0, T] \times \overline{\Omega})$  and  $D_t^{\{\tilde{k}\}}(\tilde{u} - u_0) \in C((0, T] \times \overline{\Omega})$  directly follow from the assumptions imposed on  $k$ ,  $f$  and  $u$ . Moreover,

(3.7) holds with  $k$  and  $u$  replaced by  $\tilde{k}$  and  $\tilde{u}$ . Observing the definition of  $\tilde{k}$  and the assumptions (3.2) we immediately see that  $\tilde{k}$  is decreasing. Representing  $\tilde{k}$  in the form  $\tilde{k}(t) = e^{-\sigma T} k(t) + \sigma \int_t^T e^{-\sigma s} (k(t) - k(s)) ds$  and using again (3.2) we prove the inequality  $\tilde{k} > 0$ .

Furthermore, using (3.13), the relation  $\sigma \int_0^T e^{-\sigma s} k(s) ds \rightarrow \infty$  as  $\sigma \rightarrow \infty$ , following from (3.3), as well as the inequalities  $u_0 \geq 0$ ,  $k > 0$ , we can show that there exists a sufficiently large  $\sigma$  such that the condition (3.8) is valid with  $f$  replaced by  $\tilde{f}$ .

Now we see that we can apply the first part of the proof of (i) to the problem (3.15), (3.16) to obtain  $\tilde{u} \geq 0$ .<sup>2</sup> This implies  $u \geq 0$ . The assertion (i) is completely proved.

Finally, we prove (ii). Suppose that this assertion does not hold. Then, due to the continuity of  $u$ , for some  $(t_0, x_0) \in (0, T] \times \Omega_N$ , such that  $u(t_0, x_0) = 0$ , it holds

$$\exists \delta > 0, t_2, t_3 \in (0, t_0), t_2 < t_3 : u(t, x_0) \geq \delta \text{ for } t \in (t_2, t_3). \quad (3.17)$$

We have

$$\begin{aligned} D_t^{\{k\}}[u(t, x_0) - u_0(x)] \Big|_{t=t_0} &= \lim_{\epsilon \rightarrow 0+} \frac{1}{\epsilon} \left\{ \int_0^{t_0} k(t_0 - \tau) [u(\tau, x_0) - u_0(x)] d\tau \right. \\ &\quad \left. - \int_0^{t_0 - \epsilon} k(t_0 - \epsilon - \tau) [u(\tau, x_0) - u_0(x)] d\tau \right\}. \end{aligned} \quad (3.18)$$

Let  $\epsilon < \min\{t_0 - t_3; t_2\}$ . Using (3.2), (3.17) and the relation  $u \geq 0$  we estimate the term between the brackets  $\{\}$  in (3.18):

$$\begin{aligned} &\int_0^{t_0} k(t_0 - \tau) [u(\tau, x_0) - u_0(x)] d\tau - \int_0^{t_0 - \epsilon} k(t_0 - \epsilon - \tau) [u(\tau, x_0) - u_0(x)] d\tau \\ &= \int_0^{t_0 - \epsilon} [k(t_0 - \tau) - k(t_0 - \epsilon - \tau)] u(\tau, x_0) d\tau \\ &\quad + \int_{t_0 - \epsilon}^{t_0} k(t_0 - \tau) u(\tau, x_0) d\tau - \int_{t_0 - \epsilon}^{t_0} k(\tau) d\tau u_0(x) \\ &\leq \delta \int_{t_2}^{t_3} [k(t_0 - \tau) - k(t_0 - \epsilon - \tau)] d\tau + \int_{t_0 - \epsilon}^{t_0} k(t_0 - \tau) u(\tau, x_0) d\tau \\ &= \delta \left[ \int_{t_0 - t_3}^{t_0 - t_2} k(\tau) d\tau - \int_{t_0 - t_3 - \epsilon}^{t_0 - t_2 - \epsilon} k(\tau) d\tau \right] + \int_0^\epsilon k(\tau) u(t_0 - \tau, x_0) d\tau. \end{aligned} \quad (3.19)$$

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<sup>2</sup>This is the point where we need results for more general kernel  $k(t)$  instead of  $\frac{t^{-\beta}}{\Gamma(1-\beta)}$ .

In view of (3.7) and the relation  $u(t_0, x_0) = 0$  it holds  $\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^\epsilon k(\tau)u(t_0 - \tau, x_0)d\tau = 0$ . Therefore, from (3.18) and (3.19) we get

$$\begin{aligned} D_t^{\{k\}}[u(t, x_0) - u_0(x)] \Big|_{t=t_0} &\leq \lim_{\epsilon \rightarrow 0^+} \frac{\delta}{\epsilon} \left[ \int_{t_0-t_3}^{t_0-t_2} k(\tau)d\tau - \int_{t_0-t_3-\epsilon}^{t_0-t_2-\epsilon} k(\tau)d\tau \right] \\ &= \delta \frac{d}{ds} \int_{t_0-t_3+s}^{t_0-t_2+s} k(\tau)d\tau \Big|_{s=0} = \delta[k(t_0 - t_2) - k(t_0 - t_3)] < 0, \end{aligned}$$

because  $k$  is decreasing.

On the other hand,  $(t_0, x_0)$  is a stationary local minimum point of  $u(t_0, x)$ , i.e. it holds  $\nabla u(t_0, x_0) = 0$ . This follows from the assumption  $u(t_0, x_0) = 0$  and the inequality  $u(t_0, x) \geq 0$  that holds in the neighborhood of  $x_0$  as well as from the condition  $\omega(x_0) \cdot \nabla u(t_0, x_0) = g(x_0) \geq 0$  in case II if  $x_0 \in \partial\Omega$ . Thus,  $A(x_0)u(t_0, x_0) \geq 0$ . Moreover, (3.5) with (3.4) implies  $f(u(t_0, x_0), t_0, x_0) = u(0, t_0, x_0) \geq 0$ . Consequently, the left-hand side of (3.1) is negative but the right-hand side is nonnegative at  $t = t_0$ ,  $x = x_0$ . This is a contradiction. The assertion (ii) is valid. Theorem is completely proved.  $\square$

**Corollary 1.** *Let the assumptions of Theorem 2 be satisfied. If  $f(0, t_0, x_0) > 0$  for some  $(t_0, x_0) \in (0, T] \times \Omega_N$ , then  $u(t_0, x_0) > 0$ .*

*Proof.* Due to Theorem 2, (i) we have  $u \geq 0$ . Assume that  $f(0, t_0, x_0) > 0$  for some  $(t_0, x_0) \in (0, T] \times \Omega_N$  and suppose contrary that  $u(t_0, x_0) = 0$ . Let us consider the equation (3.1) at  $t = t_0$ ,  $x = x_0$ . By Theorem 2 (ii) it holds  $u(t_0, x_0) = 0$  for all  $t \in [0, t_0]$ , thus the left-hand side of this equation is zero. Since  $(t_0, x_0)$  is a stationary local minimum point of  $u(t, \cdot)$ , it holds  $A(x_0)u(t_0, x_0) \geq 0$ . Thus, in view of the assumption  $f(0, t_0, x_0) > 0$ , the right-hand side of the equation is positive. We reached a contradiction. The assertion  $u(t_0, x_0) > 0$  is valid.  $\square$

## 4 Uniqueness results in the general case

In this section we state and prove a uniqueness theorem for IP that is the most general in the sense of the Borel measure  $\mu$ . But firstly we provide a technical lemma.

**Lemma 6.** *Let  $u \in C^\alpha([0, T]; L_s(\Omega)) \cap C([0, T] \times \overline{\Omega})$  with some  $\alpha \in (0, 1)$ ,  $s \in [1, \infty]$  and*

$$f \in \check{\mathcal{F}}_\alpha := C(\mathbb{R}; C^\alpha([0, T]; L_\infty(\Omega))) \cap C^1(\mathbb{R}; L_\infty((0, T) \times \Omega)). \quad (4.1)$$

*Then  $f(u, \cdot, \cdot) \in C^\alpha([0, T]; L_s(\Omega))$  and*

$$\|f(u, \cdot, \cdot)\|_{C^\alpha([0, T]; L_s(\Omega))} \leq \check{\mathcal{C}}_{\hat{m}, \bar{m}} (1 + \|u\|_{C^\alpha([0, T]; L_s(\Omega))}), \quad (4.2)$$

where  $\check{\mathcal{C}}_{\hat{m}, \tilde{m}}$  is a constant depending on  $\hat{m} = \min_{(t,x) \in [0,T] \times \overline{\Omega}} u(t,x)$  and  $\tilde{m} = \max_{(t,x) \in [0,T] \times \overline{\Omega}} u(t,x)$ . If, in addition,  $u \in C^\alpha([0,T]; C^\gamma(\overline{\Omega}))$  with some  $\gamma \in (0,1)$  and

$$\begin{aligned} f &\in \mathcal{F}_{\alpha,\gamma} := C(\mathbb{R}; C^\alpha([0,T]; C^\gamma(\overline{\Omega}))) \\ &\cap C^1(\mathbb{R}; C^\alpha([0,T]; L_\infty(\Omega)) \cap L_\infty((0,T); C^\gamma(\overline{\Omega}))) \\ &\cap C^2(\mathbb{R}; L_\infty((0,T) \times \Omega)) \end{aligned}$$

then  $f(u, \cdot, \cdot) \in C^\alpha([0,T]; C^\gamma(\overline{\Omega}))$  and

$$\|f(u, \cdot, \cdot)\|_{C^\alpha([0,T]; C^\gamma(\overline{\Omega}))} \leq \mathcal{C}_{\hat{m}, \tilde{m}} (1 + \|u\|_{C^\alpha([0,T]; C^\gamma(\overline{\Omega}))})^2, \quad (4.3)$$

where  $\mathcal{C}_{\hat{m}, \tilde{m}}$  is a constant depending on  $\hat{m}$  and  $\tilde{m}$ .

*Proof.* We have  $\|f(u, \cdot, \cdot)\|_{C^\alpha([0,T]; L_s(\Omega))} = \sup_{t \in (0,T)} \|f(u(t, \cdot), t, \cdot)\|_{L_s(\Omega)} + \sup_{t, \tau \in (0,T)} \frac{1}{|t-\tau|^\alpha} \|f(u(s, \cdot), s, \cdot)|_{s=\tau}^{s=t}\|_{L_s(\Omega)}$ . Splitting the function under the second sup in this formula up in the following manner:

$$f(u(s, x), s, x)|_{s=\tau}^{s=t} = f(u(t, x), s, x)|_{s=\tau}^{s=t} + \int_{u(\tau, x)}^{u(t, x)} f_w(w, \tau, x) dw \quad (4.4)$$

we deduce

$$\begin{aligned} \|f(u, \cdot, \cdot)\|_{C^\alpha([0,T]; L_s(\Omega))} &\leq \sup_{w \in [\hat{m}, \tilde{m}]} \sup_{t \in (0,T)} \|f(w, t, \cdot)\|_{L_\infty(\Omega)} (\text{meas } \Omega)^{1/s} \\ &+ \sup_{w \in [\hat{m}, \tilde{m}]} \sup_{t, \tau \in (0,T)} \frac{1}{|t-\tau|^\alpha} \|f(w, s, \cdot)|_{s=\tau}^{s=t}\|_{L_\infty(\Omega)} (\text{meas } \Omega)^{1/s} \\ &+ \sup_{w \in [\hat{m}, \tilde{m}]} \|f_w(w, \cdot, \cdot)\|_{L_\infty((0,T) \times \Omega)} \sup_{t, \tau \in (0,T)} \frac{1}{|t-\tau|^\alpha} \|u(s, \cdot)|_{s=\tau}^{s=t}\|_{L_s(\Omega)}. \end{aligned}$$

This implies  $f(u, \cdot, \cdot) \in C^\alpha([0,T]; L_s(\Omega))$  with (4.2). Further, we have

$$\begin{aligned} \|f(u(\cdot, \cdot), \cdot, \cdot)\|_{C^\alpha([0,T]; C^\gamma(\overline{\Omega}))} &= \sup_{t \in (0,T)} \left[ \sup_{x \in \Omega} |f(u(t, x), t, x)| \right. \\ &+ \sup_{x, y \in \Omega} \frac{|f(u(t, z), t, z)|_{z=y}^{z=x}}{|x-y|^\gamma} \Big] + \sup_{t, \tau \in (0,T)} \frac{1}{|t-\tau|^\alpha} \left[ \sup_{x \in \Omega} |f(u(s, x), s, x)|_{s=\tau}^{s=t} \right. \\ &+ \sup_{x, y \in \Omega} \frac{|f(u(s, z), s, z)|_{s=\tau}^{s=t} |z=y}^{z=x}}{|x-y|^\gamma} \Big]. \end{aligned}$$

Splitting the terms in this formula up by means of the relations (4.4) and the expressions

$$\begin{aligned}
f(u(t, z), t, z) \Big|_{z=y}^{z=x} &= f(u(t, x), t, z) \Big|_{z=y}^{z=x} + \int_{u(t, y)}^{u(t, x)} f_w(w, t, y) dw, \\
f(u(s, z), s, z) \Big|_{s=\tau}^{s=t} \Big|_{z=y}^{z=x} &= f(u(t, x), s, z) \Big|_{s=\tau}^{s=t} \Big|_{z=y}^{z=x} \\
&+ \int_{u(t, y)}^{u(t, x)} f_w(w, s, y) \Big|_{s=\tau}^{s=t} dw + \int_{u(\tau, x)}^{u(t, x)} f_w(w, \tau, z) \Big|_{z=y}^{z=x} dw \\
&+ \int_0^{u(t, x)-u(\tau, x)} \int_{v+u(\tau, y)}^{v+u(\tau, x)} f_{ww}(w, \tau, y) dw dv \\
&+ \int_{u(t, y)-u(\tau, y)}^{u(t, x)-u(\tau, x)} f_w(w + u(\tau, y), \tau, y) dw,
\end{aligned}$$

and estimating we obtain  $f(u, \cdot, \cdot) \in C^\alpha([0, T]; C^\gamma(\bar{\Omega}))$  with (4.3).  $\square$

**Theorem 3.** Suppose that IP has two solutions  $(z^j, u^j)$ ,  $j = 1, 2$ , such that

$$z^j \in C^\alpha(\bar{\Omega}), \quad u^j \in C^{\alpha+\beta}([0, T]; C^\alpha(\bar{\Omega})), \quad u_t^j \in C^{\bar{\alpha}}([0, T]; L_\infty(\Omega)) \quad (4.5)$$

with some  $\alpha \in (0, 1)$  and  $\bar{\alpha} \in (\max\{2\beta - 1; 0\}, 1)$ . Assume that  $a_{ij}$ ,  $a_j$ ,  $\omega$  satisfy the conditions of Theorem 1 and

$$a, a_w, b_w \in \mathcal{F}_{\alpha+\beta, \alpha}, \quad a_t, a_{ww}, a_{wt}, b_{ww}, b_{wt} \in \check{\mathcal{F}}_{\bar{\alpha}}. \quad (4.6)$$

Moreover, let

$$\begin{aligned}
a_{ww}(w, t, x)z^2(x) + b_{ww}(w, t, x), \quad a_{wt}(w, t, x)z^2(x) + b_{wt}(w, t, x) &\geq 0 \\
\text{in } [\hat{m}, \tilde{m}] \times (0, T) \times \Omega,
\end{aligned} \quad (4.7)$$

where  $\hat{m} = \min_{j=1,2} \min_{(t,x) \in [0,T] \times \bar{\Omega}} u^j(t, x)$ ,  $\tilde{m} = \max_{j=1,2} \max_{(t,x) \in [0,T] \times \bar{\Omega}} u^j(t, x)$ ,

$$u_t^1, u_t^2 \geq 0 \quad \text{in case } a_{ww} \neq 0 \text{ or } b_{ww} \neq 0, \quad (4.8)$$

$$a(u_0, 0, \cdot) = 0, \quad (4.9)$$

$$a(u^1, \cdot, \cdot) \geq 0, \quad (4.10)$$

$$\forall x \in \Omega \quad \exists \varepsilon_x > 0 : a(u^1(t, x), t, x) > 0 \text{ for } t \in (0, \varepsilon_x) \quad (4.11)$$

and the relation

$$a_w(w, t, x)z^2(x) + b_w(w, t, x) \leq \Theta \quad \text{in } [\hat{m}, \tilde{m}] \times (0, T) \times \Omega \quad (4.12)$$



holds with

$$\Theta = \sup\{\theta : (D_t^\beta - \theta)a(u^1, \cdot, \cdot) \geq 0\}. \quad (4.13)$$

Then  $z^1 = z^2$  and  $u^1 = u^2$ .

*Proof.* Without restriction of generality assume that  $\alpha \in (0, \min\{1 - \beta; \bar{\alpha} + 1 - 2\beta\})$  and  $\bar{\alpha} \in (\max\{2\beta - 1; 0\}, \beta)$ . Let us introduce the following notations:

$$q(w, t, x) = a(w, t, x)z^2(x) + b(w, t, x) \quad (4.14)$$

and  $u = u^2 - u^1$ ,  $z = z^2 - z^1$ . Then  $u$  satisfies the problem

$$\begin{aligned} D_t^\beta u(t, x) &= A(x)u(t, x) + q(u^2(t, x), t, x) - q(u^1(t, x), t, x) \\ &\quad + a(u^1(t, x), t, x)z(x), \quad (t, x) \in (0, T) \times \Omega, \end{aligned} \quad (4.15)$$

$$u(0, x) = 0, \quad x \in \Omega, \quad \mathcal{B}u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega. \quad (4.16)$$

Further, define

$$z^+ = \frac{|z| + z}{2}, \quad z^- = \frac{|z| - z}{2}.$$

Since  $z \in C^\alpha(\bar{\Omega})$ , it hold  $z^\pm \in C^\alpha(\bar{\Omega})$ . Now we consider the problems

$$\begin{aligned} D_t^\beta u^\pm(t, x) &= A(x)u^\pm(t, x) + q^0(t, x)u^\pm(t, x) \\ &\quad + a(u^1(t, x), t, x)z^\pm(x), \quad (t, x) \in (0, T) \times \Omega, \end{aligned} \quad (4.17)$$

$$u^\pm(0, x) = 0, \quad x \in \Omega, \quad \mathcal{B}u^\pm(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (4.18)$$

where

$$\begin{aligned} q^0(t, x) &= \frac{q(u^2(t, x), t, x) - q(u^1(t, x), t, x)}{u^2(t, x) - u^1(t, x)} \\ &= \int_0^1 q_w(u^1(t, x)(1 - s) + u^2(t, x)s, t, x) ds. \end{aligned} \quad (4.19)$$

Due to the assumptions (4.5), (4.6) and Lemma 6 we have

$$q^0 \in C^{\alpha+\beta}([0, T]; C^\alpha(\bar{\Omega})), \quad q_t^0 \in C^{\bar{\alpha}}([0, T]; L_\infty(\Omega)), \quad (4.20)$$

$$a(u^1, \cdot, \cdot) \in C^{\alpha+\beta}([0, T]; C^\alpha(\bar{\Omega})), \quad \frac{d}{dt}a(u^1, \cdot, \cdot) \in C^{\bar{\alpha}}([0, T]; L_\infty(\Omega)). \quad (4.21)$$

Moreover,

$$a(u^1(t, \cdot), t, \cdot)|_{t=0} = a(u_0, 0, \cdot) = 0. \quad (4.22)$$

Let  $p$  be some number such that  $\max\{\frac{n}{2}; \frac{1}{\beta-\alpha}\} < p < \infty$ ,  $p \notin \{\frac{1}{\beta} + \frac{1}{2}; \frac{2}{\beta} + 1\}$ . By (4.20), (4.21), (4.22) and Theorem 1 we see that (4.17), (4.18) have solutions satisfying

$$\begin{aligned} u^\pm &\in C^{\alpha_1+\beta}([0, T]; C^{\alpha_1}(\overline{\Omega})) \cap C^{\alpha_1}([0, T]; C^{2+\alpha_1}(\overline{\Omega})) \\ &\text{with some } \alpha_1 \in (0, \alpha] \quad \text{and} \quad u_t^\pm \in U_p. \end{aligned} \quad (4.23)$$

Moreover, taking the assumption (4.10), the relations  $z^\pm \geq 0$  and (4.18) into account and applying Theorem 2 with Remark 1 to the equation (4.17), we prove the inequality  $u^\pm \geq 0$  and the relation

$$\begin{aligned} &\text{if } u^s(t_0, x_0) \text{ with some } s \in \{+, -\} \text{ equals zero in some point} \\ &(t_0, x_0) \in (0, T] \times \Omega_N \text{ then } u^s(t, x_0) = 0 \text{ for any } t \in [0, t_0]. \end{aligned} \quad (4.24)$$

Subtracting the problems for  $u^-$  and  $u$  from the problem for  $u^+$  and observing that  $z = z^+ - z^-$ , we see that the function  $\bar{u} = u^+ - u^- - u$  solves the homogeneous equation  $D_t^\beta \bar{u} = (A + q^0)\bar{u}$  and satisfies the homogeneous initial and boundary conditions. Therefore, due to the uniqueness assertion of Theorem 1 we get  $\bar{u} = 0$ , hence

$$u = u^+ - u^-. \quad (4.25)$$

Next let us consider the problems

$$\begin{aligned} D_t^\beta v^\pm(t, x) &= A(x)v^\pm(t, x) + q^0(t, x)v^\pm(t, x) \\ &+ \varphi^\pm(t, x), \quad (t, x) \in (0, T) \times \Omega, \end{aligned} \quad (4.26)$$

$$v^\pm(0, x) = 0, \quad x \in \Omega, \quad \mathcal{B}v^\pm(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (4.27)$$

where

$$\varphi^\pm = D_t^\beta(q^0 u^\pm) - q^0 D_t^\beta u^\pm + (D_t^\beta - \Theta)a(u^1(t, x), t, x)z^\pm.$$

Analyze the formula of  $\varphi^\pm$  termwise. In view of (4.20) and  $u^\pm \in C^{\alpha_1+\beta}([0, T]; C^{\alpha_1}(\overline{\Omega}))$  it holds  $q^0 u^\pm \in C^{\alpha_1+\beta}([0, T]; C^{\alpha_1}(\overline{\Omega}))$ . Moreover,  $(q^0 u^\pm)(0, \cdot) = 0$ . Therefore, due to Lemma 2 we have  $D_t^\beta(q^0 u^\pm) \in C^{\alpha_1}([0, T]; C^{\alpha_1}(\overline{\Omega}))$  and  $D_t^\beta(q^0 u^\pm)|_{t=0} = 0$ . Similarly, we deduce  $q^0 D_t^\beta u^\pm \in C^{\alpha_1}([0, T]; C^{\alpha_1}(\overline{\Omega}))$ ,  $q^0 D_t^\beta u^\pm|_{t=0} = 0$ . In virtue of (4.21), (4.22) and Lemma 2 we get  $(D_t^\beta - \Theta)a(u^1, \cdot, \cdot)z^\pm \in C^\alpha([0, T]; C^\alpha(\overline{\Omega}))$  and  $(D_t^\beta - \Theta)a(u^1(0, \cdot), 0, \cdot)z^\pm = 0$ . Thus, we see that  $\varphi^\pm \in C^{\alpha_1}([0, T]; C^{\alpha_1}(\overline{\Omega}))$  and  $\varphi^\pm(0, \cdot) = 0$ .

Further, due to the range of  $p$  and an embedding theorem it holds  $U_p \subset C^{\bar{\alpha}}([0, T]; L_p(\Omega))$ . Thus, taking into account (4.20) and (4.23) we have  $(q^0 u^\pm)_t \in C^{\bar{\alpha}}([0, T]; L_p(\Omega))$ . Thus, taking into account the ranges of  $\alpha_1$ ,  $\alpha$ ,  $\bar{\alpha}$  and applying Lemma 2 we obtain  $D_t^\beta(q^0 u^\pm) = J_t^{1-\beta}[(q^0 u^\pm)_t] \in C^{\bar{\alpha}+1-\beta}([0, T]; L_p(\Omega)) \subset$

$C^{\alpha_1+\beta}([0, T]; L_p(\Omega))$ . Similarly we prove  $q^0 D_t^\beta u^\pm \in C^{\alpha_1+\beta}([0, T]; L_p(\Omega))$ . Finally, by the relation  $(D_t^\beta - \Theta)a(u^1, \cdot, \cdot)z^\pm = (J_t^{1-\beta} \frac{d}{dt} - \Theta)a(u^1, \cdot, \cdot)z^\pm$ , (4.21) and Lemma 2 it holds  $(D_t^\beta - \Theta)a(u^1(\cdot, \cdot), \cdot, \cdot)z^\pm \in C^{\bar{\alpha}+1-\beta}([0, T]; L_\infty(\Omega)) \subset C^{\alpha_1+\beta}([0, T]; L_p(\Omega))$ . Summing up,  $\varphi^\pm \in C^{\alpha_1+\beta}([0, T]; L_p(\Omega))$ .

We see that the assumptions of Theorem 1 are satisfied for the problems (4.26), (4.27). This implies that (4.26), (4.27) have solutions  $v^\pm \in C^{\alpha_2+\beta}([0, T]; C^{\alpha_2}(\bar{\Omega})) \cap C^{\alpha_2}([0, T]; C^{2+\alpha_2}(\bar{\Omega}))$  with some  $\alpha_2 \in (0, \alpha_1]$ .

Taking into account the smoothness properties of  $q^0$  and  $u^\pm$ , the relation  $u^\pm(0, \cdot) = 0$  and performing some elementary computations we reach the following expression:

$$(D_t^\beta(q^0 u^\pm) - q^0 D_t^\beta u^\pm)(t, x) = \int_0^t \frac{\beta(t-\tau)^{-\beta-1}}{\Gamma(1-\beta)} [q^0(t, x) - q^0(\tau, x)] u^\pm(\tau, x) d\tau.$$

By the assumptions (4.7), (4.8) and the formulas (4.14), (4.19) we see that the function  $q^0$  is nondecreasing in  $t$ . This with the proven inequalities  $u^\pm \geq 0$  implies that  $D_t^\beta(q^0 u^\pm) - q^0 D_t^\beta u^\pm \geq 0$ . Moreover, by (4.13) and  $z^\pm \geq 0$  it hold  $(D_t^\beta - \Theta)a(u^1(t, x), t, x)z^\pm \geq 0$ . Consequently, we have  $\varphi^\pm \geq 0$ . Applying Theorem 2 with Remark 1 to the equation (4.26), we obtain the inequality  $v^\pm \geq 0$  and the relation

$$\begin{aligned} &\text{if } v^s(t_0, x_0) \text{ with some } s \in \{+, -\} \text{ equals zero in some point} \\ &(t_0, x_0) \in (0, T] \times \Omega_N \text{ then } v^s(t, x_0) = 0 \text{ for any } t \in [0, t_0]. \end{aligned} \quad (4.28)$$

Next we are going to establish relations between  $v^\pm$  and  $u^\pm$ . To this end, we deduce and analyze problems for the functions  $Q^\pm = u^\pm - J_t^\beta(v^\pm + \Theta u^\pm)$ . Adding (4.17) multiplied by  $\Theta$  to (4.26), taking the operator  $J_t^\beta$  and subtracting from (4.17) after some transformations we arrive at the following equation for  $Q^\pm$ :

$$\begin{aligned} D_t^\beta Q^\pm(t, x) &= A(x)Q^\pm(t, x) + q^0(t, x)Q^\pm(t, x) + \zeta^\pm(t, x), \\ (t, x) &\in (0, T) \times \Omega, \end{aligned} \quad (4.29)$$

with the initial and boundary conditions

$$Q^\pm(0, x) = 0, \quad x \in \Omega, \quad \mathcal{B}Q^\pm(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad (4.30)$$

where

$$\begin{aligned} \zeta^\pm(t, x) &= [J_t^\beta(q^0 D_t^\beta Q^\pm) - q^0 Q^\pm](t, x) \\ &= \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} [q^0(\tau, x) - q^0(t, x)] (D_t^\beta Q^\pm)(\tau, x) d\tau \\ &= \int_0^t \left\{ \frac{(\beta-1)(t-\tau)^{\beta-2}}{\Gamma(\beta)} [q^0(\tau, x) - q^0(t, x)] \right. \\ &\quad \left. - \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} q_\tau^0(\tau, x) \right\} (J_t^{1-\beta} Q^\pm)(\tau, x) d\tau. \end{aligned}$$

Using (4.20) and the relation  $J_t^\beta J_t^{1-\beta} = J_t^1$  we deduce

$$\begin{aligned} \|\zeta^\pm(t, \cdot)\|_{L_p(\Omega)} &\leq \hat{C}_1 \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} \int_0^\tau \frac{(\tau-s)^{-\beta}}{\Gamma(1-\beta)} \|Q^\pm(s, \cdot)\|_{L_p(\Omega)} ds d\tau \\ &= \hat{C}_1 \int_0^t \|Q^\pm(\tau, \cdot)\|_{L_p(\Omega)} d\tau \end{aligned} \quad (4.31)$$

with some constant  $\hat{C}_1$ . Estimating the solution of (4.29), (4.30) by means of the technique that was used in derivation of the estimate (2.19) we obtain  $\max_{0 \leq \tau \leq t} \|Q^\pm(\tau, \cdot)\|_{L_p(\Omega)} \leq \hat{C}_2 \|\zeta^\pm\|_{L_p((0,t); L_p(\Omega))}$  for any  $t \in (0, T)$  with some constant  $\hat{C}_2$ . This with (4.31) implies

$$\left[ \max_{0 \leq \tau \leq t} \|Q^\pm(\tau, \cdot)\|_{L_p(\Omega)} \right]^p \leq \hat{C}_1 \hat{C}_2 T^p \int_0^t \left[ \max_{0 \leq \tau \leq s} \|Q^\pm(\tau, \cdot)\|_{L_p(\Omega)} \right]^p ds$$

for any  $t \in (0, T)$ . Applying Gronwall's theorem we reach the equality  $\max_{0 \leq \tau \leq t} \|Q^\pm(\tau, \cdot)\|_{L_p(\Omega)} = 0$  for any  $t \in (0, T)$ . Therefore,  $Q = u^\pm - J_t^\beta(v^\pm + \Theta u^\pm) = 0$ . This yields the following relations between  $v^\pm$  and  $u^\pm$ :

$$u^\pm - \Theta J_t^\beta u^\pm = J_t^\beta v^\pm \quad \Leftrightarrow \quad v^\pm = D_t^\beta u^\pm - \Theta u^\pm. \quad (4.32)$$

Let us return to the equation (4.17). We subtract the term  $\Theta u^\pm$  from both sides and integrate. Taking into account the right relation in (4.32) we obtain

$$\begin{aligned} \int_0^T v^\pm(t, x) d\mu &= A(x) \int_0^T u^\pm(t, x) d\mu + \int_0^T [q^0(t, x) - \Theta] u^\pm(t, x) d\mu \\ &\quad + \int_0^T a(u^1(t, x), t, x) d\mu z^\pm(x) \end{aligned} \quad (4.33)$$

for any  $x \in \Omega$ . By continuity, this relation can be extended to  $\overline{\Omega}$ . Due to the relations  $u^2 - u^1 = u^+ - u^-$  and the equality  $\int_0^T u^1(t, x) d\mu = \int_0^T u^2(t, x) d\mu$  for  $x \in \Omega$ , following from (1.6), we have

$$\int_0^T u^+(t, x) d\mu = \int_0^T u^-(t, x) d\mu \quad (4.34)$$

in  $\Omega$ . By continuity, (4.34) holds in  $x \in \overline{\Omega}$ . Define

$$x^* = \arg \max_{x \in \overline{\Omega}} \int_0^T u^\pm(t, x) d\mu. \quad (4.35)$$

Without restriction of generality we may assume that  $x^* \in \Omega_N$ . Indeed, if  $x^* \in \overline{\Omega} \setminus \Omega_N = \partial\Omega$  in case I, the vanishing boundary condition  $u^\pm|_{\partial\Omega} = 0$  implies  $\int_0^T u^\pm(t, x^*)d\mu = 0$  and since  $u^\pm \geq 0$  from (4.35) we get  $\int_0^T u^\pm(t, x)d\mu \equiv 0$ , which means that we can redefine  $x^*$  as an arbitrary point in  $\Omega$ .

According to the definitions of  $z^\pm$  it holds either  $z^+(x^*) = 0$  or  $z^-(x^*) = 0$ . Suppose that  $z^+(x^*) = 0$ . Then from (4.33) we get

$$\int_0^T v^+(t, x^*)d\mu = A(x^*) \int_0^T u^+(t, x^*)d\mu + \int_0^T [q^0(t, x^*) - \Theta] u^+(t, x^*)d\mu. \quad (4.36)$$

Let us analyze the signs of the terms in (4.36). By virtue of the proved relation  $u^+ \geq 0$  and the equality  $\mathcal{B} \int_0^T u^+(\cdot, x) = 0$ ,  $x \in \partial\Omega$ , following from (4.18),  $x^*$  is a stationary maximum point of the function  $\int_0^T u^\pm(t, x)d\mu$ . This implies that  $A(x^*) \int_0^T u^+(t, x^*)d\mu \leq 0$ . Moreover, the assumption (4.12) with (4.14), (4.19) and  $u^+ \geq 0$  implies  $\int_0^T [q^0(t, x^*) - \Theta] u^+(t, x^*)d\mu \leq 0$ . Consequently, the right-hand side of (4.36) is non-positive. Thus  $\int_0^T v^+(t, x^*)d\mu \leq 0$ . This with the proven relation  $v^+ \geq 0$  implies  $\int_0^T v^+(t, x^*)d\mu = 0$  and  $v^+(t, x^*) = 0$  for any  $t \in \text{supp}(\mu)$ . In particular,  $v^+(t^*, x^*) = 0$ , where

$$t^* = \sup \text{supp}(\mu).$$

It holds  $t^* > 0$  because of the assumption  $\text{supp}(\mu) \cap (0, T] \neq \emptyset$ . Observing the relation (4.28) we deduce  $v^+(t, x^*) = 0$  for any  $t \in [0, t^*]$ .

In view of the left equality in (4.32) we arrive at the following homogeneous Volterra equation of the 2. kind:  $(u - \Theta J_t^\beta) u^+(t, x^*) = 0$ ,  $t \in [0, t^*]$ . It has only the trivial solution

$$u^+(t, x^*) = 0, \quad t \in [0, t^*]. \quad (4.37)$$

Recall that this result is based on the supposition that  $z^+(x^*) = 0$ . Similarly we reach the equality  $u^-(t, x^*) = 0$ ,  $t \in [0, t^*]$  if we suppose  $z^-(x^*) = 0$ . Consequently, (see also (4.34)) we obtain the relation  $\int_0^T u^\pm(t, x^*)d\mu = 0$ .

Recall that  $x^*$  is the maximum point of the non-negative function  $\int_0^T u^\pm(t, x)d\mu$ . Therefore, it holds  $\int_0^T u^\pm(\cdot, x)d\mu = 0$  for any  $x \in \Omega$ . Since  $u^\pm \geq 0$ , we get  $u^\pm(t, x) = 0$  for any  $t \in \text{supp}(\mu)$  and  $x \in \Omega$ , in particular  $u^\pm(t^*, x) = 0$  for any  $x \in \Omega$ . Applying (4.24) we deduce

$$u^\pm(t, x) = 0 \quad \text{for any } t \in [0, t^*] \text{ and } x \in \Omega.$$

Due to this relation, the equation (4.17) reduces to the form

$$a(u^1(t, x), t, x) z^\pm(x) = 0$$

in the cylinder  $(t, x) \in (0, t^*) \times \Omega$ . In view of the assumption (4.11) we obtain  $z^\pm(x) = 0$  in  $\Omega$ . Observing that  $z^2 - z^1 = z = z^+ - z^-$  we reach the assertion  $z^1 = z^2$ . Finally, due to  $z^\pm = 0$ , the linear problem (4.17), (4.18) is homogeneous. By virtue of the uniqueness statement of Theorem 1 we get  $u^\pm = 0$ . This in view of  $u^2 - u^1 = u = u^+ - u^-$  yields  $u^1 = u^2$ . Theorem is completely proved.  $\square$

From the proved theorem we infer the following uniqueness result for a linear inverse problem.

**Corollary 2.** *Let  $a(w, t, x) = a(t, x)$  and  $b(w, t, x) = b^1(t, x)w + b^2(t, x)$ . Suppose that IP has two solutions  $(z^j, u^j)$ ,  $j = 1, 2$ , satisfying (4.5) with some  $\alpha \in (0, 1)$  and  $\bar{\alpha} \in (\max\{2\beta - 1; 0\}, 1)$ . Assume that  $a_{ij}$ ,  $a_j$ ,  $\omega$  fulfill the conditions of Theorem 1,  $a, b^1 \in C^{\alpha+\beta}([0, T]; C^\alpha(\Omega))$ ,  $a_t, b_t^1 \in C^{\bar{\alpha}}([0, T], L_\infty(\Omega))$ ,  $a, b_t^1 \geq 0$ ,  $a(0, \cdot) = 0$ ,*

$$\forall x \in \Omega \quad \exists \varepsilon_x > 0 : a(t, x) > 0 \text{ for } t \in (0, \varepsilon_x) \quad (4.38)$$

and

$$b^1 \leq \Theta = \sup\{\theta : (D_t^\beta - \theta)a \geq 0\}. \quad (4.39)$$

Then  $z^1 = z^2$  and  $u^1 = u^2$ .

## 5 Uniqueness in a particular case

In case the measure  $\mu$  has a special form, the uniqueness can be proved under lower regularity assumptions and the cone condition (4.7) and the restriction (4.8) dropped. Let us consider the following particular case:

$$d\mu = \varkappa(t)dt, \text{ where } dt \text{ is the Lebesgue measure and } \varkappa \geq 0, \varkappa \neq 0. \quad (5.1)$$

**Theorem 4.** *Let (5.1) hold, where  $\varkappa \in W_s^1(0, T)$  with some  $s > \frac{1}{1-\beta}$ . Suppose that IP has two solutions  $(z^j, u^j)$ ,  $j = 1, 2$ , such that*

$$z^j \in C^\alpha(\overline{\Omega}), \quad u^j \in C^{\alpha+\beta}([0, T]; L_\infty(\Omega)) \cap C^\alpha([0, T]; C^\alpha(\overline{\Omega})) \quad (5.2)$$

with some  $\alpha \in (0, 1)$ . Assume that  $a_{ij}$ ,  $a_j$ ,  $\omega$  satisfy the conditions of Theorem 1 and

$$a, a_w, b_w \in \check{\mathcal{F}}_{\alpha+\beta} \cap \mathcal{F}_{\alpha, \alpha}. \quad (5.3)$$

Moreover, let the relations (4.9), (4.10), (4.11) hold and the inequality (4.12) be valid with some  $\Theta < \hat{\theta} = \sup\{\theta : \kappa - \theta\varkappa \geq 0\}$ , where

$$\kappa(t) = \frac{1}{\Gamma(1-\beta)} \left[ (T-t)^{-\beta} \varkappa(T) - \int_t^T (\tau-t)^{-\beta} \varkappa'(\tau) d\tau \right]. \quad (5.4)$$

Then  $z^1 = z^2$  and  $u^1 = u^2$ .

*Proof.* The beginning of the proof coincides with the section of the proof of Theorem 3 that starts with the formula (4.14) and ends with the relation (4.19). From (4.19) on we continue in a different manner. Due to assumptions of the theorem and Lemma 6, it hold

$$q^0, a(u^1, \cdot, \cdot) \in C^{\alpha+\beta}([0, T]; L_\infty(\Omega)) \cap C^\alpha([0, T]; C^\alpha(\overline{\Omega})). \quad (5.5)$$

Taking also the relation  $a(u^1(t, \cdot), t, \cdot)|_{t=0} = a(u_0, 0, \cdot) = 0$  into account and using Theorem 1 we come to the conclusion that (4.17), (4.18) have unique solutions  $u^\pm \in C^{\alpha_1+\beta}([0, T]; C^{\alpha_1}(\overline{\Omega})) \cap C^{\alpha_1}([0, T]; C^{2+\alpha_1}(\overline{\Omega}))$  with some  $\alpha_1 \in (0, 1 - \beta)$  and  $D_t^\beta u^\pm \in C^{\alpha_1}([0, T]; C^{\alpha_1}(\overline{\Omega}))$ . Like in the proof of Theorem 3, we deduce the relations  $u^\pm \geq 0$ , (4.24) and the equality (4.25).

Let us subtract the term  $\Theta u^\pm$  from the left and right-hand side of (4.17) and integrate (4.17) in the Lebesgue sense with the weight  $\varkappa$ . Integrating by parts at the left-hand side and performing some simple computations we obtain

$$\begin{aligned} \int_0^T [\kappa(t) - \Theta \varkappa(t)] u^\pm(t, x) dt &= A(x) \int_0^T u^\pm(t, x) \varkappa(t) dt \\ &+ \int_0^T [q^0(t, x) - \Theta] u^\pm(t, x) \varkappa(t) dt + \int_0^T a(u^1(t, x), t, x) \varkappa(t) dt z^\pm(x) \end{aligned} \quad (5.6)$$

for any  $x \in \overline{\Omega}$ . By the relations  $u^2 - u^1 = u^+ - u^-$ , the equality  $\int_0^T u^1(t, x) \varkappa(t) dt = \int_0^T u^2(t, x) \varkappa(t) dt$  for  $x \in \Omega$  and continuity we have

$$\int_0^T u^+(t, x) \varkappa(t) dt = \int_0^T u^-(t, x) \varkappa(t) dt \quad (5.7)$$

in  $\overline{\Omega}$ . Define

$$x^* = \arg \max_{x \in \overline{\Omega}} \int_0^T u^\pm(t, x) \varkappa(t) dt. \quad (5.8)$$

Without restriction of generality we may assume that  $x^* \in \Omega_N$ .

It holds either  $z^+(x^*) = 0$  or  $z^-(x^*) = 0$ . Let  $z^+(x^*) = 0$ . Then from (5.6) we get

$$\begin{aligned} \int_0^T [\kappa(t) - \Theta \varkappa(t)] u^+(t, x^*) dt &= A(x) \int_0^T u^+(t, x^*) \varkappa(t) dt \\ &+ \int_0^T [q^0(t, x^*) - \Theta] u^+(t, x^*) \varkappa(t) dt. \end{aligned} \quad (5.9)$$

Arguing similarly as in the case of the equation (4.36), we show that the right-hand side of (5.9) is non-positive. Thus, we obtain

$$\int_0^T [\kappa(t) - \Theta \varkappa(t)] u^+(t, x^*) dt \leq 0. \quad (5.10)$$

According to the definitions of  $\Theta$  and  $\hat{\theta}$  we have  $\kappa - \Theta \varkappa = \kappa - \hat{\theta} \varkappa + (\hat{\theta} - \Theta) \varkappa \geq (\hat{\theta} - \Theta) \varkappa$ , where  $\hat{\theta} - \Theta > 0$ . Thus,  $\kappa - \Theta \varkappa \geq 0$  and due to the relation  $\text{supp}(\varkappa) \cap (0, T] \neq \emptyset$ , following from the assumptions imposed on  $\varkappa$ , there exists  $t_1 \in (0, T]$  such that  $\kappa(t_1) - \Theta \varkappa(t_1) > 0$ . Therefore, observing also the property  $u^+ \geq 0$ , from (5.10) we get  $[\kappa - \Theta \varkappa] u^+(\cdot, x^*) = 0$  and  $u^+(t_1, x^*) = 0$ . Now the relation (4.24) implies  $u^+(t, x^*) = 0$ ,  $t \in [0, t_1]$ .

The rest of the proof repeats the part of the proof of Theorem 3 that follows the formula (4.37) (with  $t_1$  instead of  $t^*$ ).  $\square$

**Corollary 3.** *Let (5.1) hold, where  $\varkappa \in W_s^1(0, T)$  with some  $s > \frac{1}{1-\beta}$  and  $a(w, t, x) = a(t, x)$ ,  $b(w, t, x) = b^1(t, x)w + b^2(t, x)$ . Suppose that IP has two solutions  $(z^j, u^j)$ ,  $j = 1, 2$ , satisfying (5.2) with some  $\alpha \in (0, 1)$ . Assume that  $a_{ij}$ ,  $a_j$ ,  $\omega$  satisfy the conditions of Theorem 1,  $a, b^1 \in C^\alpha([0, T]; C^\alpha(\bar{\Omega}))$ ,  $a \geq 0$ ,  $a(0, \cdot) = 0$ , (4.38) is valid and  $b^1 \leq \Theta < \hat{\theta}$  with  $\hat{\theta}$  defined in Theorem 4. Then  $z^1 = z^2$  and  $u^1 = u^2$ .*

**Remark 2.** In case  $\varkappa = 1$  we have  $\kappa(t) = \frac{(T-t)^{-\beta}}{\Gamma(1-\beta)}$  and  $\hat{\theta} = \frac{T^{-\beta}}{\Gamma(1-\beta)}$ .

## 6 Additional remarks

The aim of this section is to interpret the conditions (4.8), (4.10), (4.11) in a suitable way and estimate the quantity  $\Theta$  occurring in the conditions (4.12), (4.39) from below.

**Theorem 5.** *Let  $u$  solve the problem (1.1), (1.2), (1.3) with  $f(w, t, x) = a(w, t, x)z(x) + b(w, t, x)$ . Assume  $u \in C([0, T] \times \bar{\Omega})$ ,  $u, u_{x_j}, u_{x_i x_j} \in C^{\beta''}([0, T]; L_1(\Omega))$  with some  $\beta'' > 1 - \beta$ ,  $u_t \in C^{\beta'}([0, T]; C(\bar{\Omega}))$  with some  $\beta' > \beta$ ,  $u_{tx_j}, u_{tx_i x_j} \in C((0, T] \times \bar{\Omega})$ ,  $a_{ij}, a_j \in C(\bar{\Omega})$ ,  $\omega \in (C(\partial\Omega))^n$ ,  $a, b \in C(\mathbb{R}; C^{\beta''}([0, T]; L_\infty(\Omega)))$ ,  $a_w, b_w, a_t, b_t \in C(\mathbb{R} \times [0, T] \times \bar{\Omega})$ ,  $a(u_0, 0, \cdot) = 0$ ,  $Au_0 + b(u_0, 0, \cdot) = 0$ ,  $g_t \geq 0$  and*

$$a_t(u, \cdot, \cdot)z + b_t(u, \cdot, \cdot) \geq 0. \quad (6.1)$$

*Then  $u_t \geq 0$ . If in addition*

$$a(u, \cdot, \cdot), a_w(u, \cdot, \cdot), a_t(u, \cdot, \cdot) \geq 0, \quad (6.2)$$

*then*

$$\Theta = \sup\{\theta : (D_t^\beta - \theta)a(u, \cdot, \cdot) \geq 0\} \geq \frac{T^{-\beta}}{\Gamma(1-\beta)}. \quad (6.3)$$



*Proof.* Firstly, let us prove  $u_t \geq 0$ . The equation for  $u$  can be rewritten in the form  $\int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} u_\tau(\tau, x) d\tau = A(x)u(t, x) + a(u(t, x), t, x)z(x) + b(u(t, x), t, x)$ . The left hand side of this equation equals  $J_t^{1-\beta} u_t$ . Therefore, applying the operator  $D_t^{1-\beta}$  to this equation we get  $u_t = D_t^{1-\beta} [Au + a(u, \cdot, \cdot)z + b(u, \cdot, \cdot)]$ . By virtue of the assumptions of the theorem and Lemma 6 we have  $Au + a(u, \cdot, \cdot)z + b(u, \cdot, \cdot) \in C^{\beta''}([0, T]; L_1(\Omega))$  and  $[Au + a(u, \cdot, \cdot)z + b(u, \cdot, \cdot)]|_{t=0} = 0$ . Applying Lemma 2 we get  $D_t^{1-\beta} [Au + a(u, \cdot, \cdot)z + b(u, \cdot, \cdot)]|_{t=0} = 0$ . Thus,  $u_t(0, \cdot) = 0$ . Now from the equation of  $u$  we deduce the following equation for  $u_t$ :  $D_t^\beta [u_t - u_t(0, \cdot)] = Au_t + [a_w(u, \cdot, \cdot)z + b_w(u, \cdot, \cdot)]u_t + a_t(u, \cdot, \cdot)z + b_t(u, \cdot, \cdot)$ . Applying Theorem 2 to this equation we reach the assertion  $u_t \geq 0$ .

Secondly, let us prove (6.3). Note that

$$\begin{aligned} D_t^\beta a(u(t, x), t, x) &= \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} \frac{d}{d\tau} a(u(\tau, x), \tau, x) d\tau \\ &= \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} \frac{d}{d\tau} [a(u(\tau, x), \tau, x) - a(u(t, x), t, x)] d\tau \\ &= \int_0^t \frac{\beta(t-\tau)^{-\beta-1}}{\Gamma(1-\beta)} [a(u(t, x), t, x) - a(u(\tau, x), \tau, x)] d\tau \\ &\quad + a(u(t, x), t, x) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \end{aligned}$$

Observing (6.2) and  $u_t \geq 0$  we deduce (6.3). Proof is complete.  $\square$

Verification of the relations (4.11), (6.2) and interpretation of the cone conditions (4.7), (4.12), (6.1) for  $z$  requires a priori information about the bounds of  $u$ . Most simple way to remove  $u$  from these conditions is to assume them to hold for any  $w \in \mathbb{R}$ . For instance, sufficient conditions for (6.2) and (4.11) are

$$a(w, t, x), a_w(w, t, x), a_t(w, t, x) \geq 0 \quad (6.4)$$

in  $\mathbb{R} \times (0, T) \times \Omega$  and

$$\forall x \in \Omega \quad \exists \varepsilon_x > 0 : a(w, t, x) > 0 \quad (6.5)$$

for  $w \in \mathbb{R}$  and  $t \in (0, \varepsilon_x)$ , respectively. However, this is too restrictive, because in many practical cases the nonlinearity function  $a$  is not positive and monotone for all  $w$  (see the examples below). The proved positivity principle makes possible to restrict these conditions for positive or negative values of  $w$ .

Suppose that the assumptions of Theorem 2 are satisfied for  $a_{ij}, a_j, \omega, u$  and  $a, b \in C(\mathbb{R} \times [0, T] \times \overline{\Omega})$ ,  $z \in C(\overline{\Omega})$ . Let us point out two cases.

1. If  $u_0, g \geq 0$  and (3.5) holds with  $f = az + b$ , then by Theorem 2 we have  $u \geq 0$ . This means that for the condition (6.2) to hold it is sufficient to assume that the relations (6.4) are satisfied in  $(0, M) \times (0, T) \times \Omega$ , where  $M \in (0, \infty]$  is some *a priori* known upper bound of  $u$ . If in addition  $\forall x \in \Omega \exists \delta_x > 0 : f(0, t, x) > 0$  in  $(0, \delta_x)$  then by Corollary 1 the inequality  $u(t, x) > 0$  is valid in  $(0, \delta_x) \times \Omega$ . Therefore, for the condition (4.11) to hold it is sufficient to assume that (6.5) is satisfied for any  $w \in (0, M)$  and  $t \in (0, \varepsilon_x)$  and  $\varepsilon_x$  is taken so that  $\varepsilon_x \leq \delta_x$ .
2. The statements can be reformulated for negative  $u$ , too. To this end, Theorem 2 and Corollary 1 have to be applied to the problem for  $-u$ . If  $u_0, g \leq 0$  and (3.5) holds with  $f(w, t, x) = -a(-w, t, x)z(x) - b(-w, t, x)$ , then  $u \leq 0$ . For the condition (6.2) to hold it is sufficient to assume that the relations (6.4) are satisfied in  $(m, 0) \times (0, T) \times \Omega$ , where  $m \in [-\infty, 0)$  is an *a priori* known lower bound of  $u$ . If in addition  $\forall x \in \Omega \exists \delta_x > 0 : f(0, t, x) > 0$  in  $(0, \delta_x)$  then by Corollary 1 the inequality  $u(t, x) < 0$  is valid in  $(0, \delta_x) \times \Omega$ . For the condition (4.11) to hold it is sufficient to assume that (6.5) is satisfied for any  $w \in (m, 0)$  and  $t \in (0, \varepsilon_x)$  and  $\varepsilon_x \leq \delta_x$ .

Clearly, similar statements can be formulated for the conditions (4.7), (4.12), (6.1), as well.

Finally, we consider the satisfaction of the cone conditions in Theorems 3 and 4 in case of some particular equations occurring in applications. Let  $a(w, t, x) = a(w)$  and  $b(w, t, x) = b(t, x)$ . This means that  $za$  is an inhomogeneous reaction term and  $b$  is a source term. Let us choose the following three examples:

- (1) linear reaction (or potential)  $a(w, t, x) = w$  [11];
- (2) nonlinear reaction  $a(w, t, x) = w(1 - \frac{w}{W})$  occurring in the Fisher equation [25, 26];
- (3) nonlinear reaction  $a(w, t, x) = w^2(1 - \frac{w}{W})$  in the Zeldovich equation [25].

Here  $W$  is a given positive constant.

Let  $u_0 = 0, g \geq 0$ . Then  $u \geq 0$ . In context of Theorem 3 we assume also  $g_t \geq 0$  that yields  $u_t \geq 0$ . Moreover, let  $b(0, x) = 0$  and  $b(t, x) > 0$  for  $t \in (0, \delta)$  and some  $\delta > 0$ . (Alternatively we could set  $b = 0$ , but then the additional restriction  $u(t, x) > 0$  for  $t \in (0, \delta)$  is needed). By means of rather elementary computations, from (4.7), (4.12), (6.2), (6.3) and Remark 2 we deduce the following sufficient conditions in the form of direct inequalities for  $z$  and  $u$  for assumptions of Theorems 3 and 4.

The cone conditions of Theorem 4 with  $\varkappa = 1$  are satisfied provided

$$\begin{aligned}
z^2 &\leq \frac{T^{-\beta}}{\Gamma(1-\beta)} \text{ in case (1);} \\
z^2 &\leq \frac{T^{-\beta}}{\Gamma(1-\beta)}, u^j \leq \frac{W}{2}, j = 1, 2, \text{ in case (2);} \\
z^2 &\leq \frac{3T^{-\beta}}{\Gamma(1-\beta)W}, u^j \leq \frac{2W}{3}, j = 1, 2, \text{ in case (3).}
\end{aligned}$$

The cone conditions of Theorem 3 hold if

$$\begin{aligned}
z^2 &\leq \frac{T^{-\beta}}{\Gamma(1-\beta)} \text{ in case (1);} \\
z^2 &\leq 0, u^j \leq \frac{W}{2}, j = 1, 2, \text{ in case (2);} \\
z^2 &\leq \frac{3T^{-\beta}}{\Gamma(1-\beta)W}, u^j \leq \frac{W}{3}, j = 1, 2, \text{ in case (3).}
\end{aligned}$$

Theorem 3 fails in case (2) for positive  $z^2$ . In any of the other cases, the smaller  $T$ , the less restrictive the inequalities for  $z^2$  and  $u^j$  are. (For  $u^j$ , this follows from the homogeneous initial condition.)

## Acknowledgement

The research was supported by Estonian Research Council Personal Research Grant PUT568 and Institutional Research Grant IUT33-24.

## Appendix

In this appendix we prove local existence and global uniqueness theorems for the problem (1.1) - (1.3).

**Theorem A1.** *Assume that  $a_{ij}$ ,  $a_j$ ,  $\omega$  and  $p$  satisfy the assumptions of Theorem 1. Let there exist a function  $\hat{u} \in U_p \cap C([0, T] \times \overline{\Omega})$  that satisfies the conditions<sup>3</sup>*

$$\hat{u}(0, x) = u_0(x), \quad x \in \Omega, \quad \mathcal{B}\hat{u}(t, x) = g(t, x), \quad (t, x) \in (0, T) \times \Omega$$

and fulfills the relation  $\Psi := D_t^\beta(\hat{u} - u_0) - A\hat{u} - f(\hat{u}, \cdot, \cdot) \in C([0, T]; C_*(\overline{\Omega}))$ , where  $C_*(\overline{\Omega}) = \{z \in C(\overline{\Omega}) : z|_{\partial\Omega} = 0\}$  in case I and  $C_*(\overline{\Omega}) = C(\overline{\Omega})$  in case II. Concerning  $f$ , we assume that  $f \in C(\mathbb{R} \times [0, T] \times \overline{\Omega})$  and

$$\begin{aligned}
&\exists K, \varrho > 0 : |f(\hat{u} + w^1, t, x) - f(\hat{u} + w^2, t, x)| \leq K|w^1 - w^2| \\
&\forall w^1, w^2 \in [-\varrho, \varrho], t \in [0, T], x \in \overline{\Omega}.
\end{aligned} \tag{7.1}$$

If  $T$  is smaller than a certain constant depending on  $A$ ,  $\beta$ ,  $\Psi$ ,  $K$  and  $\varrho$ , then the problem (1.1) - (1.3) has a solution in the space  $U_p \cap C([0, T] \times \overline{\Omega})$ .

*Proof.* Firstly, let us show that the following inequality is valid:

$$\|J_t^\beta y\|_{C^\alpha([0, T]; X)} \leq \frac{1}{\beta\Gamma(\beta)}(T^\beta + 2T^{\beta-\alpha})\|y\|_{C([0, T]; X)} \tag{7.2}$$

for any  $y \in C([0, T]; X)$ , where  $X$  is a Banach space and  $\alpha \in (0, \beta)$ . Indeed,

$$\begin{aligned}
\Gamma(\beta)\|J_t^\beta y\|_{C^\alpha([0, T]; X)} &= \sup_{0 < t < T} \left\| \int_0^t (t - \tau)^{\beta-1} y(\tau) d\tau \right\| \\
&\quad + \sup_{0 < s < t < T} \frac{1}{(t-s)^\alpha} \left\| \int_0^t (t - \tau)^{\beta-1} y(\tau) d\tau - \int_0^s (s - \tau)^{\beta-1} y(\tau) d\tau \right\| \\
&\leq (I_1 + I_2 + I_3)\|y\|_{C([0, T]; X)},
\end{aligned}$$

---

<sup>3</sup>The space  $U_p$  was defined in (2.5).

where

$$\begin{aligned}
I_1 &= \sup_{0 < t < T} \int_0^t (t - \tau)^{\beta-1} d\tau = \sup_{0 < t < T} \int_0^t \tau^{\beta-1} d\tau = \frac{T^\beta}{\beta}, \\
I_2 &= \sup_{0 < s < t < T} \frac{1}{(t-s)^\alpha} \int_0^s |(t - \tau)^{\beta-1} - (s - \tau)^{\beta-1}| d\tau = \sup_{0 < s < t < T} \frac{1}{(t-s)^\alpha} \\
&\times \left( \int_0^s \tau^{\beta-1} d\tau - \int_{t-s}^t \tau^{\beta-1} d\tau \right) \leq \sup_{0 < s < t < T} \frac{1}{(t-s)^\alpha} \int_0^{t-s} \tau^{\beta-1} d\tau = \frac{T^{\beta-\alpha}}{\beta}, \\
I_3 &= \sup_{0 < s < t < T} \frac{1}{(t-s)^\alpha} \int_s^t (t - \tau)^{\beta-1} d\tau = \sup_{0 < s < t < T} \frac{1}{(t-s)^\alpha} \int_0^{t-s} \tau^{\beta-1} d\tau = \frac{T^{\beta-\alpha}}{\beta}.
\end{aligned}$$

This implies (7.2).

Making use of the change of variables  $u = \hat{u} + v$ , the problem (1.1) - (1.3) for  $u \in U_p \cap C([0, T] \times \overline{\Omega})$  is reduced to the following problem for  $v \in U_p \cap C([0, T] \times \overline{\Omega})$ :

$$\begin{aligned}
D_t^\beta v(t, x) &= A(x)v(t, x) + F(v(t, x), t, x), \quad t \in (0, T), \quad x \in \Omega, \\
v(0, x) &= 0, \quad x \in \Omega, \quad \mathcal{B}v(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega,
\end{aligned} \tag{7.3}$$

where  $F(v, t, x) = f(\hat{u} + v, t, x) - f(\hat{u}, t, x) - \Psi(t, x)$ . Due to Lemma 1, (7.3) is in  $U_p \cap C([0, T] \times \overline{\Omega})$  equivalent to the problem

$$\begin{aligned}
v(t, x) &= A(x)J_t^\beta v(t, x) + J_t^\beta F(v(t, x), t, x), \quad t \in (0, T), \quad x \in \Omega, \\
v(0, x) &= 0, \quad x \in \Omega, \quad \mathcal{B}v(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega.
\end{aligned} \tag{7.4}$$

On the other hand, there exists  $\xi \in \mathbb{R}$  such that the operator  $\mathcal{A} = A + \xi$  satisfies the assumptions of Lemma 4 in the space  $C_*(\overline{\Omega})$  with the domain  $D(\mathcal{A}) = \{z : z \in W_q^2(\Omega) \forall q \in (1, \infty), \mathcal{A}z \in C_*(\overline{\Omega}), \mathcal{B}z|_{\partial\Omega} = 0\}$  (see [18], Sect. 3.1.5). Let us consider the following operator equation:

$$v = \mathcal{F}v \quad \text{with} \quad \mathcal{F} = \mathcal{Q}_\alpha J_t^\beta [F(v, \cdot, \cdot) - \xi v] \tag{7.5}$$

in the ball  $B_\varrho = \{v : v \in C([0, T]; C_*(\overline{\Omega})), \|v\|_{C([0, T]; C_*(\overline{\Omega}))} \leq \varrho\}$ , where  $\alpha = \beta/2$  and  $\mathcal{Q}_\alpha$  is the operator defined in Lemma 4. Using Lemma 4, the definition of  $F$ , (7.1) and (7.2) we obtain for any  $v \in B_\varrho$  the estimate

$$\begin{aligned}
\|\mathcal{F}v\|_{C([0, T]; C_*(\overline{\Omega}))} &\leq \|\mathcal{F}v\|_{C^\alpha([0, T]; C_*(\overline{\Omega}))} \leq \frac{\|\mathcal{Q}_\alpha\|}{\beta\Gamma(\beta)} (T^\beta + 2T^{\beta/2}) \\
&\times [(K + |\xi|)\|v\|_{C([0, T]; C_*(\overline{\Omega}))} + \|\Psi\|_{C([0, T]; C_*(\overline{\Omega}))}] \\
&\leq \frac{\|\mathcal{Q}_\alpha\|}{\beta\Gamma(\beta)} (T^\beta + 2T^{\beta/2}) [(K + |\xi|)\varrho + \|\Psi\|_{C([0, T]; C_*(\overline{\Omega}))}].
\end{aligned}$$

From this estimate we see that if  $T$  is smaller than a certain constant  $T_1$  depending on  $\|\mathcal{Q}_\alpha\|$ ,  $\beta$ ,  $K$ ,  $\xi$ ,  $\varrho$  and  $\Psi$ , then the inequality  $\|\mathcal{F}v\|_{C([0, T]; C_*(\overline{\Omega}))} \leq \varrho$  is valid.

This means that for  $T < T_1$  the operator  $\mathcal{F}$  maps  $B_\varrho$  into  $B_\varrho$ . Similarly, for any  $v^1, v^2 \in B_\varrho$  we deduce the estimate

$$\|\mathcal{F}v^1 - \mathcal{F}v^2\|_{C([0,T];C_*(\overline{\Omega}))} \leq \kappa \|v^1 - v^2\|_{C([0,T];C_*(\overline{\Omega}))}$$

with  $\kappa = \frac{\|\mathcal{Q}_\alpha\|}{\beta\Gamma(\beta)}(T^\beta + 2T^{\beta/2})(K + |\xi|)$ . There exists  $T_2$  depending on  $\|\mathcal{Q}_\alpha\|$ ,  $\beta$ ,  $K$ ,  $\xi$  such that if  $T < T_2$  then  $\kappa < 1$ . We have shown that  $\mathcal{F}$  is a contraction in  $B_\varrho$  provided  $T < T_3 = \min\{T_1; T_2\}$ . Thus, (7.5) has a unique solution in  $B_\varrho$  for  $T < T_3$ .

Further, for this solution we have the relation  $F(v, \cdot, \cdot) - \xi v \in C([0, T]; C_*(\overline{\Omega}))$ . Therefore, in view of (7.2), it holds  $\varphi = J_t^\beta[F(v, \cdot, \cdot) - \xi v] \in C^\alpha([0, T]; C_*(\overline{\Omega}))$ . By virtue of Lemma 4, the operator equation (7.5) is equivalent to (7.4) and its solution  $v$  belongs to  $C^\alpha([0, T]; C_*(\overline{\Omega}))$ . On the other hand, since  $[F(v, \cdot, \cdot) - \xi v] \in L_p((0, T); L_p(\Omega))$ , due to Lemma 1 we get  $J_t^\beta[F(v, \cdot, \cdot) - \xi v] \in H_p^\beta([0, T]; L_p(\Omega))$  and Lemma 2 implies that the solution  $v$  of (7.4) belongs to  $U_p$ . Summing up,  $v \in U_p \cap C([0, T]; C_*(\overline{\Omega}))$  and in view of the assumptions imposed on  $\hat{u}$  we obtain  $u = \hat{u} + v \in U_p \cap C([0, T] \times \overline{\Omega})$ . The proof is complete.  $\square$

**Remark A1.** Let us briefly discuss simple possibilities to construct functions  $\hat{u}$  satisfying the conditions of Theorem A1. Firstly, consider the case II. Assume  $u_0 \in C^{2+l}(\overline{\Omega})$ ,  $g \in C^{1+l, \frac{1+l}{2}}([0, T] \times \partial\Omega)$  for some  $l \in (0, 1)$ , where  $C^{\gamma, \frac{\gamma}{2}}$  is the anisotropic Hölder space (see [15]), and  $\mathcal{B}u_0(x) = g(0, x)$  for  $x \in \partial\Omega$ . Define  $\hat{u}$  as the solution of  $\hat{u}_t - A\hat{u} = 0$  subject to (1.2) and (1.3). Then  $\hat{u}$ ,  $\hat{u}_t$  and  $A\hat{u}$  are continuous in  $[0, T] \times \overline{\Omega}$  [15], hence  $D_t^\beta(\hat{u} - u_0) = J_t^{1-\beta}\hat{u}_t$  and  $f(\hat{u}, \cdot, \cdot)$  are also continuous. We immediately obtain  $\Psi \in C([0, T]; C_*(\overline{\Omega}))$ . In case I, let  $u_0 \in C^{2+l}(\overline{\Omega})$ ,  $g \in C^{2+l, 1+\frac{l}{2}}([0, T] \times \partial\Omega)$  for some  $l \in (0, 1)$  and  $u_0(x) = g(0, x)$  for  $x \in \partial\Omega$ . Moreover, assume  $f(g, \cdot, \cdot) \in C^{l, \frac{l}{2}}([0, T] \times \partial\Omega)$ . Then the function  $\zeta = g_t - D_t^\beta(g - u_0) + f(g, \cdot, \cdot) = g_t - J_t^\beta g_t + f(g, \cdot, \cdot)$  belongs to  $C^{l, \frac{l}{2}}([0, T] \times \partial\Omega)$ . Let us continue  $\zeta$  to a function  $C^{l, \frac{l}{2}}([0, T] \times \overline{\Omega})$  and assume additionally  $Au_0(x) + f(g(0, x), 0, x) = 0$  for  $x \in \partial\Omega$ . Define  $\hat{u}$  as the solution of  $\hat{u}_t - A\hat{u} = \zeta$  subject to (1.2) and (1.3). Again,  $\hat{u}$ ,  $\hat{u}_t$  and  $A\hat{u}$  are continuous, hence  $\Psi$  is continuous. Due to the choice of  $\zeta$  we have  $\Psi(t, x) = 0$  for  $x \in \partial\Omega$ . Thus,  $\Psi \in C([0, T]; C_*(\overline{\Omega}))$ .

**Theorem A2.** Assume that the assumptions of Theorem 2 are satisfied for  $a_{ij}$ ,  $a_j$ ,  $\omega$  and (3.4) hold. Suppose that (1.1) - (1.3) has two solutions  $u^1$  and  $u^2$  satisfying the relations  $u^l \in C^{\beta'}([0, T]; C(\overline{\Omega}))$ ,  $l = 1, 2$ , for some  $\beta' > \beta$  and  $u_{x_j}^l, u_{x_i, x_j}^l \in C((0, T] \times \overline{\Omega})$ ,  $l = 1, 2$ . Moreover, let the inequality  $|f(u^1(t, x), t, x) - f(u^2(t, x), t, x)| \leq M|u^1(t, x) - u^2(t, x)|$  be valid for  $t \in [0, T]$ ,  $x \in \overline{\Omega}$  with some  $M > 0$ . Then  $u^1 = u^2$ .

*Proof.* The difference  $u = u^1 - u^2$  solves the problem

$$\begin{aligned} D_t^\beta u(t, x) &= A(x)u(t, x) + G(u(t, x), t, x), \quad t \in (0, T), \quad x \in \Omega, \\ u(0, x) &= 0, \quad x \in \Omega, \quad \mathcal{B}u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \end{aligned} \quad (7.6)$$

with linear with respect to  $w$  function  $G(w, t, x) = \frac{f(u^1(t, x), t, x) - f(u^2(t, x), t, x)}{u^1(t, x) - u^2(t, x)}w$ . Applying Theorem 2 to problems of  $u$  and  $-u$  we obtain  $u \geq 0$  and  $u \leq 0$ . This implies the assertion of the theorem.  $\square$

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